Boundary Conditions for Limited Area Models

Ming-Cheng Shiue (National Chiao Tung University)

Joint work with
Arthur Bousquet, Qingshan Chen, Madalina Petcu,
Jacques Laminie, Roger Temam, Joseph Tribbia
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4. Summary
Limited Area Models (LAMs) are often used to achieve high resolution over a region of interest.
Figure 1: Taiwan. [Online Image] Availability http://www.google.com/, December 21, 2010.
Limited Area Models

Figure 2: Coastal Flow, San Pedro, California.[Online Image] Availability
Challenge: lateral boundary conditions (LBCs)
No physical laws can provide natural boundary conditions at the lateral boundary. Furthermore, for computational purposes, we want the lateral boundary conditions to be transparent.

Figure 4: Transparent property
**Difficulty:**

- **On the computational side**
  Errors at the lateral boundary will propagate and advect into the modeled domain and have a major impact inside the domain.

- **On the mathematical side**
  Oliger and Sundstrom, 1978 showed the ill-posedness of a class of equations of geophysical fluid mechanics supplemented with any set of local boundary conditions. This class of equations includes the inviscid Primitive equations and the Shallow Water equations in the multi-layer case.
Background

Difficulty:

- **On the computational side**
  Errors at the lateral boundary will propagate and advect into the modeled domain and have a major impact inside the domain.

- **On the mathematical side**
  Oliger and Sundstrom, 1978 showed the **ill-posedness** of a class of equations of geophysical fluid mechanics supplemented with any set of local boundary conditions. This class of equations includes the **inviscid Primitive equations** and the **Shallow Water equations** in the **multi-layer case**.
Boundary conditions for Shallow Water equations

The one-dimensional transport equation:

\[ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0. \]

\( a > 0 \)

\( u|_{x=0} \) given

\( x=0 \) inflow boundary

\( x=L \) outflow boundary

\( \) No imposition

**Figure 5: Boundary Conditions**
The One Dimensional Linearized Shallow Water Equations:

\[
\begin{align*}
    u_t + u_0 u_x + gh_x &= 0, \\
    h_t + u_0 h_x + h_0 u_x &= 0.
\end{align*}
\]

\[
\frac{\partial}{\partial t} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} u_0 - \sqrt{gh_0} \\ 0 \\ 0 \\ u_0 + \sqrt{gh_0} \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0,
\]

where

\[
\begin{align*}
    \xi_1 &= u - \sqrt{\frac{g}{h_0}} h, \\
    \xi_2 &= u + \sqrt{\frac{g}{h_0}} h.
\end{align*}
\]
Boundary conditions for Shallow Water equations

The One Dimensional Linearized Shallow Water Equations:

\[
\begin{align*}
&u_t + u_0 u_x + gh_x = 0, \\
&h_t + u_0 h_x + h_0 u_x = 0.
\end{align*}
\]

where

\[
\begin{align*}
\xi_1 &= u - \sqrt{\frac{g}{h_0}} h, \\
\xi_2 &= u + \sqrt{\frac{g}{h_0}} h.
\end{align*}
\]
Subcritical flow \( (u_0 - \sqrt{gh_0} < 0) \)

\( \xi_1 \) (outflow) \hspace{1cm} \xi_2 \) (inflow) \hspace{1cm} \xi_1 \) (inflow) \hspace{1cm} \xi_2 \) (outflow)

\( \xi_2 \big|_{x=0} = 0 \)

\( \xi_1 \big|_{x=L} = 0 \)

Figure 6:
Boundary conditions for Shallow Water equations

Supercritical flow \((u_0 - \sqrt{gh_0} > 0)\)

- \(\xi_1\) \(x=0\) = 0
- \(\xi_2\) \(x=0\) = 0

Figure 7:
Boundary conditions for Shallow Water equations

Equations:

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} - fv &= -g \frac{\partial B}{\partial x}, \quad x \in (0, L), \ t > 0, \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + fu &= 0, \\
\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + \frac{\partial u}{\partial x} h &= 0.
\end{align*}
\]

Initial conditions:

\[u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), \ h(x, 0) = h_0(x), \ 0 < x < L.\]
Consider the Shallow Water equations linearized around the simple uniform flow:

\[ \bar{u} = u_0, \quad \bar{v} = v_0, \quad \text{and} \quad \bar{h} = h_0. \]

We set

\[
\begin{aligned}
    u &= \bar{u} + \tilde{u}, \\
    v &= \bar{v} + \tilde{v}, \\
    h &= \bar{h} + \tilde{h}.
\end{aligned}
\]

\[
\begin{align*}
    \tilde{u}_t + u_0 \tilde{u}_x + g\tilde{h}_x - f\tilde{v} &= 0, \\
    \tilde{v}_t + u_0 \tilde{v}_x + f\tilde{u} &= 0, \\
    \tilde{h}_t + u_0 \tilde{h}_x + h_0 \tilde{u}_x &= 0.
\end{align*}
\]

(LSWEs)
Consider perturbed energy:

\[
\frac{d}{dt} \int_0^L (\tilde{u}^2 + \tilde{v}^2 + \frac{g}{h_0} \tilde{h}^2) \, dx = I(0, t) - I(L, t),
\]

where

\[
l(x, t) = u_0(\tilde{u}^2(x, t) + \tilde{v}^2(x, t) + \frac{g}{h_0} \tilde{h}^2(x, t)) + 2g\tilde{u}(x, t)\tilde{h}(x, t),
\]

\[
= \begin{pmatrix} \tilde{u} & \tilde{v} & \frac{\sqrt{g}}{\sqrt{h_0}} \tilde{h} \end{pmatrix} \begin{pmatrix} u_0 & 0 & \sqrt{gh_0} \\ 0 & u_0 & 0 \\ \frac{\sqrt{gh_0}}{\sqrt{h_0}} & 0 & u_0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{g}}{\sqrt{h_0}} \tilde{h} \\ \frac{\sqrt{g}}{\sqrt{h_0}} \tilde{h} \end{pmatrix},
\]

\[
= (u_0 - \sqrt{gh_0})\tilde{\alpha}^2(x, t) + u_0\tilde{\beta}^2(x, t) + (u_0 + \sqrt{gh_0})\tilde{\gamma}^2(x, t),
\]

where
Boundary conditions for Shallow Water equations

\[
\begin{align*}
\tilde{\alpha}(x, t) &= \frac{\tilde{u}(x, t)}{\sqrt{2}} - \sqrt{\frac{g}{2h_0}} \tilde{h}(x, t), \\
\tilde{\beta}(x, t) &= \tilde{v}(x, t), \\
\tilde{\gamma}(x, t) &= \frac{\tilde{u}(x, t)}{\sqrt{2}} + \sqrt{\frac{g}{2h_0}} \tilde{h}(x, t).
\end{align*}
\]

Flow type:
Subcritical flows: \( u_0 - \sqrt{gh_0} < 0 \)
Supercritical flows: \( u_0 - \sqrt{gh_0} > 0 \)

For subcritical flows:
\[
\begin{align*}
\tilde{u}(L, t) - \sqrt{\frac{g}{h_0}} \tilde{h}(L, t) &= 0, \\
\tilde{v}(0, t) &= 0, \\
\tilde{u}(0, t) + \sqrt{\frac{g}{h_0}} \tilde{h}(0, t) &= 0.
\end{align*}
\]

For supercritical flows:
\[
\begin{align*}
\tilde{u}(0, t) - \sqrt{\frac{g}{h_0}} \tilde{h}(0, t) &= 0, \\
\tilde{v}(0, t) &= 0, \\
\tilde{u}(0, t) + \sqrt{\frac{g}{h_0}} \tilde{h}(0, t) &= 0.
\end{align*}
\]
Linear type of characteristic boundary conditions: For subcritical flows:

\[
\begin{align*}
\alpha_1(L, t) &= u(L, t) - \sqrt{\frac{g}{h_0}} h(L, t) = u_0 - \sqrt{gh_0}, \\
\beta_1(0, t) &= v(0, t) = v_0, \\
\gamma_1(0, t) &= u(0, t) + \sqrt{\frac{g}{h_0}} h(0, t) = u_0 + \sqrt{gh_0}.
\end{align*}
\] (2)

For supercritical flows:

\[
\begin{align*}
\alpha_1(0, t) &= u(0, t) - \sqrt{\frac{g}{h_0}} h(0, t) = u_0 - \sqrt{gh_0}, \\
\beta_1(0, t) &= v(0, t) = v_0, \\
\gamma_1(0, t) &= u(0, t) + \sqrt{\frac{g}{h_0}} h(0, t) = u_0 + \sqrt{gh_0}.
\end{align*}
\] (3)
Nonlinear type of characteristic boundary conditions: Inspired by the theoretical work presented in the book of Benzoni and Serre, we consider the boundary conditions

For subcritical flows:

\[
\begin{align*}
\alpha_2(L, t) &= \frac{u(L, t)}{2} - \sqrt{gh(L, t)} = \frac{u_0}{2} - \sqrt{gh_0}, \\
\beta_2(0, t) &= v(0, t) = v_0, \\
\gamma_2(0, t) &= \frac{u(0, t)}{2} + \sqrt{gh(0, t)} = \frac{u_0}{2} + \sqrt{gh_0}.
\end{align*}
\]  

(4)

For supercritical flows:

\[
\begin{align*}
\alpha_2(0, t) &= \frac{u(0, t)}{2} - \sqrt{gh(0, t)} = \frac{u_0}{2} - \sqrt{gh_0}, \\
\beta_2(0, t) &= v(0, t) = v_0, \\
\gamma_2(0, t) &= \frac{u(0, t)}{2} + \sqrt{gh(0, t)} = \frac{u_0}{2} + \sqrt{gh_0}.
\end{align*}
\]  

(5)
Equations for $\alpha_2, \beta_2, \gamma_2$:

\[
\begin{align*}
\frac{\partial \alpha_2}{\partial t} + \left(\frac{3\alpha_2 + \gamma_2}{2}\right) \frac{\partial \alpha_2}{\partial x} &= \frac{f \beta_2 - g B_x}{2}, \\
\frac{\partial \beta_2}{\partial t} + (\alpha_2 + \gamma_2) \frac{\partial \beta_2}{\partial x} &= -f(\alpha_2 + \gamma_2), \\
\frac{\partial \gamma_2}{\partial t} + \left(\frac{\alpha_2 + 3\gamma_2}{2}\right) \frac{\partial \gamma_2}{\partial x} &= \frac{f \beta_2 - g B_x}{2}.
\end{align*}
\] (6)
Boundary conditions for Shallow Water equations

The well-posed issues:

Local existence and uniqueness for the system and $\nu \equiv 0$ with spatial periodic boundary conditions (see Rakotoson-Temam-Tribbia(2008))

$$u(0, t) = u(L, t), \ h(0, t) = h(L, t).$$

Local existence and uniqueness for the system and $\nu \equiv 0$ with Dirichlet boundary condition on $u$ only (see Petcu-Temam(2010))

$$u(0, t) = u(L, t) = 0.$$

Local existence and uniqueness for the system with a set of transparent boundary conditions (see Petcu-Temam(2011) and Huang-Petcu-Temam(2011)).
We rewrite the SWEs in conservative form as follows:

\[
\frac{\partial}{\partial t} U + \frac{\partial}{\partial x} F(U) = S(U, t, x),
\]

where

\[
U = \begin{pmatrix} uh \\ vh \\ h \end{pmatrix}, \quad F(U) = \begin{pmatrix} hu^2 + \frac{1}{2} gh^2 \\ uvh \\ uh \end{pmatrix}, \quad S = \begin{pmatrix} fvh - gh \frac{\partial}{\partial x} B \\ -fuh \\ 0 \end{pmatrix}.
\]
Numerical schemes: semidiscrete central-upwind method

Finite volume method:

\[ \bar{U}_j(t) := \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} U(t, x) \, dx. \]  \hfill (9)

Discretized system:

\[ \frac{d}{dt} \bar{U}_j(t) + \frac{F(U(t, x_{j+\frac{1}{2}})) - F(U(t, x_{j-\frac{1}{2}}))}{\Delta x} = \frac{\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} S(U(t, x), t, x) \, dx}{\Delta x}. \]  \hfill (10)

Issues:
The approximation of the fluxes \( F(U) \)
The approximation of the source terms \( S(U) \)
The approximations of the fluxes $F(U)$ at the points $x = x_{j + \frac{1}{2}}$ are given by

$$F(U(t, x_{j + \frac{1}{2}})) \approx F_{j + \frac{1}{2}}(t),$$

(11)

where

$$F_{j + \frac{1}{2}}(t) := \frac{a^+_{j + \frac{1}{2}} F(U^-_{j + \frac{1}{2}}) - a^-_{j + \frac{1}{2}} F(U^+_{j + \frac{1}{2}})}{a^+_{j + \frac{1}{2}} - a^-_{j + \frac{1}{2}}} + \frac{a^+_{j + \frac{1}{2}} a^-_{j + \frac{1}{2}}}{a^+_{j + \frac{1}{2}} - a^-_{j + \frac{1}{2}}} [U^+_{j + \frac{1}{2}} - U^-_{j + \frac{1}{2}}].$$

(12)

Here $U^+_{j + \frac{1}{2}} := p_{j + 1}(t, x_{j + \frac{1}{2}})$ and $U^-_{j + \frac{1}{2}} := p_j(t, x_{j + \frac{1}{2}})$, where $p_j(t, x)$ are non-oscillatory linear polynomial reconstructions

$$p_j(t, x) = \bar{U}_j + s_m(t)(x - x_j),$$

where
\( s_m(t) := \minmod(\theta \frac{\bar{U}_{j+1} - \bar{U}_j}{\Delta x}, \frac{\bar{U}_{j+1} - \bar{U}_{j-1}}{2\Delta x}, \theta \frac{\bar{U}_j - \bar{U}_{j-1}}{\Delta x}) \), \hspace{1cm} (13)

with

\[
\minmod(x_1, x_2, \cdots) := \begin{cases} 
\min(x_i), & \text{if } x_i > 0 \ \forall \ i, \\
\max(x_i), & \text{if } x_i < 0 \ \forall \ i, \\
0, & \text{otherwise},
\end{cases}
\hspace{1cm} (14)
\]

and \( \theta \in [1, 2] \). Finally, the one-sided local speeds of propagation \( a_{j+\frac{1}{2}}^\pm \) are given by

\[
a_{j+\frac{1}{2}}^+ := \max(\lambda_{\max}(\frac{\partial F}{\partial U}(U_{j+\frac{1}{2}}^+)), \lambda_{\max}(\frac{\partial F}{\partial U}(U_{j+\frac{1}{2}}^-)), 0),
\]

\[
a_{j+\frac{1}{2}}^- := \min(\lambda_{\min}(\frac{\partial F}{\partial U}(U_{j+\frac{1}{2}}^+)), \lambda_{\min}(\frac{\partial F}{\partial U}(U_{j+\frac{1}{2}}^-)), 0),
\hspace{1cm} (15)
\]

where \( \lambda_{\max}(\frac{\partial F}{\partial U}(\bar{U})) \) and \( \lambda_{\min}(\frac{\partial F}{\partial U}(\bar{U})) \) are the largest and smallest eigenvalues of the differential \( \frac{\partial F}{\partial U} \) at the point \( U = \bar{U} \).
The approximation of the source terms

Using the midpoint rule for the spatial integral:

\[ \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} S(U(t, x), t, x) \, dx \approx S(U(t, x_j), t, x_j). \]  

(16)
ODE system:

\[ \frac{d}{dt} \bar{U}_j(t) + \frac{F_{j+\frac{1}{2}}(t) - F_{j-\frac{1}{2}}(t)}{\Delta x} = S_j(t), \quad (17) \]

where

\[ S_j(t) := S(U(t, x_j), t, x_j). \]

RK2 method:

\[
\begin{align*}
\frac{U_j^* - U_j^n}{\Delta t} &= -\frac{F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n}{\Delta x} + S_j^n, \\
\frac{U_j^{**} - U_j^*}{\Delta t} &= -\frac{F_{j+\frac{1}{2}}^* - F_{j-\frac{1}{2}}^*}{\Delta x} + S_j^*, \\
U_j^{n+1} &= \frac{U_j^n + U_j^{**}}{2}.
\end{align*}
\quad (18)
\]
Nonlinear type of characteristic boundary conditions:
For subcritical flows, $x = 0$,

$$\frac{\alpha_{2,1}^{n+1} - \alpha_{2,1}^n}{\Delta t} + \left( \frac{3\alpha_{2,1}^n + \gamma_{2,1}^n}{2} \right) \frac{\alpha_{2,2}^{n+1} - \alpha_{2,1}^{n+1}}{\Delta x} = \frac{f\beta_{2,1}^n - gB_{x,1}^n}{2}. \quad (19)$$

For subcritical flows, $x = L$,

$$\frac{\beta_{2,M+1}^{n+1} - \beta_{2,M+1}^n}{\Delta t} + \left( \alpha_{2,M+1}^n + \gamma_{2,M+1}^n \right) \frac{\beta_{2,M+1}^{n+1} - \beta_{2,M}^{n+1}}{\Delta x} = -f(\alpha_{2,M+1}^n + \gamma_{2,M+1}^n), \quad (20)$$

and

$$\frac{\gamma_{2,M+1}^{n+1} - \gamma_{2,M+1}^n}{\Delta t} + \left( \frac{\alpha_{2,M+1}^n + 3\gamma_{2,M+1}^n}{2} \right) \frac{\gamma_{2,M+1}^{n+1} - \gamma_{2,M}^{n+1}}{\Delta x} = \frac{f\beta_{2,M+1}^n - gB_{x,M+1}^n}{2}. \quad (21)$$
For supercritical flows, \( x = L \),

\[
\frac{\alpha_{2,M+1}^{n+1} - \alpha_{2,M+1}^n}{\Delta t} + \left( \frac{3\alpha_{2,M+1}^n + \gamma_{2,M+1}^n}{2} \right) \frac{\alpha_{2,M+1}^{n+1} - \alpha_{2,M}^{n+1}}{\Delta x} = \frac{f\beta_{2,M+1}^n - gB_{x,M+1}^n}{2}. \tag{22}
\]

along with (20) and (21).
Boundary conditions for Shallow Water equations

Numerical examples:

**Subcritical flows:**

**Initial Conditions:**

\[
\begin{align*}
  u(x,0) &= u_0, \
  v(x,0) &= v_0, \
  h(x,0) &= \begin{cases} 
  h_0 - B(x) + \epsilon h_0, & \text{if } \kappa \leq x \leq 2\kappa, \\
  h_0 - B(x), & \text{otherwise}.
\end{cases}
\end{align*}
\]

Here \( u_0 = 0 \text{ m/s} \), \( v_0 = 0 \text{ m/s} \), \( h_0 = 10^4 \text{ m} \) and \( \epsilon = 0.2 \). The bottom topography consists of one hump,

\[
B(x) = \begin{cases} 
  \frac{\delta}{2} + \frac{\delta}{2} \cos\left( \frac{\pi(x - \frac{L}{2})}{\kappa} \right), & \text{if } |x - \frac{L}{2}| \leq \kappa, \\
  0, & \text{otherwise},
\end{cases}
\]

where \( \delta = 5 \times 10^3 \text{ m} \), \( \kappa = L/10 \), and \( L = 10^6 \text{ m} \).
Subcritical Flows: nonlinear type boundary conditions

Simulation:
Supercritical Flows: **Initial Conditions:**

\[ u(x, 0) = u_0, \ v(x, 0) = v_0, \ h(x, 0) = h_0 - B(x). \]

Here \( u_0 = 450 \text{ m/s} \), \( v_0 = 0 \text{ m/s} \), \( h_0 = 5 \times 10^3 \text{ m} \). The bottom topography consists of one hump,

\[
B(x) = \begin{cases} 
\frac{\delta}{2} + \frac{\delta}{2} \cos \left( \frac{\pi (x - \frac{L}{2})}{\kappa} \right), & \text{if } |x - \frac{L}{2}| \leq \kappa, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \delta = 2.5 \times 10^3 \text{ m} \), \( \kappa = L/10 \), and \( L = 10^6 \text{ m} \).
Supercritical Flows: nonlinear type boundary conditions:

Simulation:
The 3D inviscid full Primitive Equations:

\[
\begin{align*}
\frac{\partial \tilde{v}}{\partial t} + (\tilde{v} \cdot \nabla)\tilde{v} + \tilde{w} \frac{\partial \tilde{v}}{\partial z} + f k \times \tilde{v} + \frac{1}{\rho_0} \nabla \tilde{p} &= 0, \text{ (Momentum equation)} \\
\frac{\partial \tilde{p}}{\partial z} &= -\tilde{\rho} g, \text{ (Hydrostatic equation)} \\
\nabla \cdot \tilde{v} + \frac{\partial \tilde{w}}{\partial z} &= 0, \text{ (Continuity equation)} \\
\frac{\partial \tilde{T}}{\partial t} + (\tilde{v} \cdot \nabla)\tilde{T} + \tilde{w} \frac{\partial \tilde{T}}{\partial z} &= 0, \text{ (Thermodynamics equation)} \\
\tilde{\rho} &= \rho_0 (1 - \alpha (\tilde{T} - T_0)), \text{ (Equation of states)}
\end{align*}
\]

\(\tilde{v}: = (\tilde{u}, \tilde{v})\) the horizontal velocity \(\tilde{w}: \) the vertical velocity

\(\tilde{\rho}: \) the density \(\tilde{\rho}: \) the pressure

\(\nabla: \) the horizontal gradient operator \(\tilde{T}: \) the temperature
Linearization around the simple uniform stratified flow

The domain under consideration is $\mathcal{M} = \mathcal{M}' \times (0, -L_3)$, where $\mathcal{M}' = (0, L_1) \times (0, L_2)$.

Linearization around the simple uniform stratified flow:

\[
\begin{align*}
\bar{u} &= \bar{U}_0, \quad \bar{v} = 0, \quad \bar{w} = 0, \\
\bar{T} &= \frac{N^2}{\alpha g} z, \quad \bar{\rho} = -\frac{\rho_0 N^2}{g} z, \quad \frac{d\bar{P}(z)}{dz} = -(\rho_0 + \bar{\rho})g,
\end{align*}
\]

where $\bar{U}_0$, $\rho_0$, and $T_0$ are positive constants and we introduce the Brunt–Väisälä (buoyancy) frequency

\[N^2 = \frac{g}{\rho_0} \frac{d\bar{\rho}}{dz}.
\]

In this work, we assume that $N$ is a positive constant. Setting:

\[
\begin{align*}
\tilde{u} &= \bar{u} + u(x, y, z, t), \\
\tilde{v} &= \bar{v} + v = v(x, y, z, t), \\
\tilde{w} &= \bar{w} + w = w(x, y, z, t), \\
\tilde{T} &= T_0 + \bar{T}(z) + T(x, y, z, t), \\
\tilde{\rho} &= \rho_0 + \bar{\rho}(z) + \rho(x, y, z, t),
\end{align*}
\]
The domain under consideration is $\mathcal{M} = \mathcal{M}' \times (0, -L_3)$, where $\mathcal{M}' = (0, L_1) \times (0, L_2)$.

Linearization around the simple uniform stratified flow:

$$
\bar{u} = \bar{U}_0, \quad \bar{v} = 0, \quad \bar{w} = 0,
$$

$$
\bar{T} = \frac{N_2^2}{\alpha g} z, \quad \bar{\rho} = -\frac{\rho_0 N_2^2}{g} z, \quad \frac{d\bar{P}(z)}{dz} = -(\rho_0 + \bar{\rho})g,
$$

where $\bar{U}_0$, $\rho_0$, and $T_0$ are positive constants and we introduce the Brunt–Väisälä (buoyancy) frequency

$$
N_2 = \frac{g \ d\bar{\rho}}{\rho_0 \ dz}.
$$

In this work, we assume that $N$ is a positive constant. Setting:

$$
\tilde{u} = \bar{u} + u(x, y, z, t), \quad \tilde{T} = T_0 + \bar{T}(z) + T(x, y, z, t),
$$

$$
\tilde{v} = \bar{v} + v = v(x, y, z, t), \quad \tilde{\rho} = \rho_0 + \bar{\rho}(z) + \rho(x, y, z, t),
$$

$$
\tilde{w} = \bar{w} + w = w(x, y, z, t), \quad \tilde{\rho} = \rho_0 + \bar{\rho}(z) + \rho(x, y, z, t),
$$

Ming-Cheng Shiue
The domain under consideration is $\mathcal{M} = \mathcal{M}' \times (0, -L_3)$, where $\mathcal{M}' = (0, L_1) \times (0, L_2)$.

Linearization around the simple uniform stratified flow:

$$
\begin{align*}
\bar{u} &= \bar{U}_0, \quad \bar{v} = 0, \quad \bar{w} = 0, \\
\bar{T} &= \frac{N^2}{\alpha g} z, \quad \bar{\rho} = -\frac{\rho_0 N^2}{g} z, \quad \frac{d\bar{P}(z)}{dz} = -(\rho_0 + \bar{\rho})g,
\end{align*}
$$

where $\bar{U}_0$, $\rho_0$, and $T_0$ are positive constants and we introduce the Brunt–Väisälä (buoyancy) frequency

$$N^2 = \frac{g \ d\bar{\rho}}{\rho_0 \ dz}.$$

In this work, we assume that $N$ is a positive constant. Setting:

$$
\begin{align*}
\tilde{u} &= \bar{u} + u(x, y, z, t), \quad \tilde{T} = T_0 + \bar{T}(z) + T(x, y, z, t), \\
\tilde{v} &= \bar{v} + v = v(x, y, z, t), \quad \tilde{\rho} = \rho_0 + \bar{\rho}(z) + \rho(x, y, z, t), \\
\tilde{w} &= \bar{w} + w = w(x, y, z, t), \quad \tilde{\rho} = \rho_0 + \bar{\rho}(z) + p(x, y, z, t).
\end{align*}
$$
We have the following equations for $u$, $v$, $w$, $\phi = p/\rho_0$, and $\psi = \phi_z = \alpha gT$:

$$
\begin{aligned}
&u_t + \bar{U}_0 u_x - f v + \phi_x + B(u, v, w; u) = 0, \\
v_t + \bar{U}_0 v_x + f u + \phi_y + f u + B(u, v, w; v) + f \bar{U}_0 = 0, \\
&\phi_z = -\frac{\rho}{\rho_0} g = \psi, \\
&u_x + v_y + w_z = 0, \\
&\psi_t + \bar{U}_0 \psi_x + N^2 w + B(u, v, w; \psi) = 0,
\end{aligned}
$$

(24)

where

$$
B(u, v, w; \theta) = u\theta_x + v\theta_y + w\theta_z, \text{ for } \theta = u, v, \text{ or } \psi.
$$
The Linearized system

\[
\begin{cases}
    u_t + \bar{U}_0 u_x - f v + \phi_x = 0, \\
    v_t + \bar{U}_0 v_x + f u + \phi_y = 0, \\
    \phi_z = \psi, \\
    u_x + v_y + w_z = 0, \\
    \psi_t + \bar{U}_0 \psi_x + N^2 w = 0.
\end{cases}
\] (25)

Separation of variables:

\[
\begin{cases}
    u(x, y, z, t) = \mathcal{U}(z) \hat{u}(x, y, t), \\
    v(x, y, z, t) = \mathcal{V}(z) \hat{v}(x, y, t), \\
    \psi(x, y, z, t) = \Psi(z) \hat{\psi}(x, y, t), \\
    w(x, y, z, t) = \mathcal{W}(z) \hat{w}(x, y, t), \\
    \phi(x, y, z, t) = \Phi(z) \hat{u}(x, y, t),
\end{cases}
\]

For simplicity, we take

\[\mathcal{U} = \mathcal{V} = \Phi, \mathcal{W} = \Psi.\]
The Linearized system

\[
\begin{align*}
    u_t + \bar{U}_0 u_x - f v + \phi_x &= 0, \\
    v_t + \bar{U}_0 v_x + f u + \phi_y &= 0, \\
    \phi_z &= \psi, \\
    u_x + v_y + w_z &= 0, \\
    \psi_t + \bar{U}_0 \psi_x + N^2 w &= 0.
\end{align*}
\]  
(25)

Separation of variables:

\[
\begin{align*}
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    v(x, y, z, t) &= \mathcal{V}(z) \hat{v}(x, y, t), \\
    \psi(x, y, z, t) &= \Psi(z) \hat{\psi}(x, y, t), \\
    w(x, y, z, t) &= \mathcal{W}(z) \hat{w}(x, y, t), \\
    \phi(x, y, z, t) &= \Phi(z) \hat{u}(x, y, t),
\end{align*}
\]

For simplicity, we take

\[
\mathcal{U} = \mathcal{V} = \Phi, \quad \mathcal{W} = \Psi.
\]

Ming-Cheng Shiue
The Linearized system

\[
\begin{aligned}
&u_t + \bar{U}_0 u_x - f v + \phi_x = 0, \\
v_t + \bar{U}_0 v_x + f u + \phi_y = 0, \\
\phi_z = \psi, \\
u_x + v_y + w_z = 0, \\
\psi_t + \bar{U}_0 \psi_x + N^2 w = 0.
\end{aligned}
\] (25)

Separation of variables:

\[
\begin{aligned}
u(x, y, z, t) &= \mathcal{U}(z) \hat{u}(x, y, t), \\
v(x, y, z, t) &= \mathcal{V}(z) \hat{v}(x, y, t), \\
\psi(x, y, z, t) &= \mathcal{\Psi}(z) \hat{\psi}(x, y, t), \\
w(x, y, z, t) &= \mathcal{W}(z) \hat{w}(x, y, t), \\
\phi(x, y, z, t) &= \Phi(z) \hat{u}(x, y, t),
\end{aligned}
\]

For simplicity, we take

\[
\mathcal{U} = \mathcal{V} = \Phi, \quad \mathcal{W} = \Psi.
\]
The Linearized system

From the third and fourth equations in (25), we find that the corresponding Sturm-Liouville problems are as follows:

\[ u'' + \lambda^2 u = 0, \quad w'' + \lambda^2 w = 0, \]

with the following natural boundary conditions:

\[ u'(0) = u'(-L_3) = 0, \quad w(0) = w(-L_3) = 0. \]

The boundary conditions come from the following top and bottom boundary conditions:

\[ w(z = 0) = w(z = -L_3) = 0. \]
The Linearized system

From the third and fourth equations in (25), we find that the corresponding Sturm-Liouville problems are as follows:

\[ U'' + \lambda^2 U = 0, \quad W'' + \lambda^2 W = 0, \]

with the following natural boundary conditions:

\[ U'(0) = U'(-L_3) = 0, \quad W(0) = W(-L_3) = 0. \]

The boundary conditions come from the following top and bottom boundary conditions:

\[ w(z = 0) = w(z = -L_3) = 0. \]
Normal Mode Expansion in $z$

We look for the general solutions of (24) in the form:

\[
\begin{align*}
(u, v, \phi) &= \sum_{n \geq 0} U_n(z)(u_n, v_n, \phi_n)(x, y, t), \\
(w, \psi) &= \sum_{n \geq 1} W_n(z)(w_n, \psi_n)(x, y, t).
\end{align*}
\]  

(26)

where $U_n(z)$ and $W_n(z)$ are the eigenfunctions of the Sturm-Liouville problem associated with linearized equations of (24) and

\[
\begin{align*}
\lambda_n &= \frac{n\pi}{L_3}, \\
W_n(z) &= \sqrt{\frac{2}{L_3}} \sin(\lambda_n z), U_n(z) = \sqrt{\frac{2}{L_3}} \cos(\lambda_n z), n \geq 1, \\
U_0(z) &= \frac{1}{\sqrt{L_3}}.
\end{align*}
\]  

(27)
The barotropic mode

For \( n = 0 \), we find that

\[
\begin{align*}
\frac{\partial u_0}{\partial t} + \bar{U}_0 \frac{\partial u_0}{\partial x} + \frac{\partial \phi_0}{\partial x} - fv_0 + \int_{-L_3}^{0} B(u, v, w; u) U_0(z) \, dz &= 0, \\
\frac{\partial v_0}{\partial t} + \bar{U}_0 \frac{\partial v_0}{\partial y} + \frac{\partial \phi_0}{\partial y} + fu_0 + \int_{-L_3}^{0} B(u, v, w; v) U_0(z) \, dz + f\bar{U}_0 \sqrt{L_3} &= 0, \\
\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} &= 0, \\
\psi_0 = w_0 &= 0,
\end{align*}
\]

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For $n \geq 1$, we find that

\[
\begin{align*}
\frac{\partial u_n}{\partial t} + \bar{U}_0 \frac{\partial u_n}{\partial x} - f v_n + \frac{\partial \phi_n}{\partial x} + \int_{-L_3}^{0} B(u, v, w; u) U_n(z) \, dz &= 0, \\
\frac{\partial v_n}{\partial t} + \bar{U}_0 \frac{\partial v_n}{\partial x} + f u_n + \frac{\partial \phi_n}{\partial y} + \int_{-L_3}^{0} B(u, v, w; v) U_n(z) \, dz &= 0, \\
\frac{\partial \psi_n}{\partial t} + \bar{U}_0 \frac{\partial \psi_n}{\partial x} + N^2 w_n + \int_{-L_3}^{0} B(u, v, w; \psi) W_n(z) \, dz &= 0, \\
\phi_n &= -\frac{1}{\lambda_n} \psi_n, \quad w_n = -\frac{1}{\lambda_n} \left( \frac{\partial u_n}{\partial x} + \frac{\partial v_n}{\partial y} \right). \tag{29}
\end{align*}
\]
Barotropic flows

Equations for barotropic flows without coupling other modes:

\[
\begin{aligned}
\mathbf{v}_t + \bar{U}_0 \mathbf{v}_x + f \mathbf{k} \times \mathbf{v} + \nabla \phi_0 + \frac{1}{\sqrt{L_3}} (\mathbf{v} \cdot \nabla) \mathbf{v} &= F, \\
\text{div}\mathbf{v} &= 0.
\end{aligned}
\]  

(30)

Here,
\[
\mathbf{v} = (u_0, v_0)^T
\]
Boundary conditions for barotropic flows

We propose the following boundary conditions:

\[
\begin{align*}
    u_0 &= 0, \quad \text{at } x = 0, \ L_1. \\
    v_0 &= 0, \quad \text{at } x = 0, \text{ and } y = 0, \ L_2.
\end{align*}
\]  

(31)

**Figure 11:** Boundary conditions for barotropic flows

**Remark 1**

- *The initial boundary problem is not classical.*
Numerical schemes for the zero mode

Let $\Delta t = T/K$, $v^k \approx v(x, y, k\Delta t)$, and $v^{k+\frac{1}{2}}$ represents an intermediate value between $v^k$ and $v^{k+1}$, etc.

First Step:

$$\begin{cases}
\frac{v^{k+\frac{1}{2}} - v^k}{\Delta t} + \bar{U}_0 v^{k+\frac{1}{2}} + f_k \times v^k + \nabla \phi^k_0 + G^k_0 = 0, \\
v^{k+\frac{1}{2}}|_{x=0} = 0,
\end{cases}$$

(32)

Here

$$G^k_0 = \left( \begin{array}{c}
\int_0^0 B(u^k, v^k, w^k; u^k) u_0(z) \, dz \\
\int_{-L_3}^0 B(u^k, v^k, w^k; v^k) u_0(z) \, dz + f \bar{U}_0 \sqrt{L_3} \end{array} \right).$$

(33)

Second Step: (projection method)

$$\begin{cases}
\frac{v^{k+1} - v^{k+\frac{1}{2}}}{\Delta t} + \nabla (\phi^{k+1}_0 - \phi^k_0) = 0, \\
\nabla \cdot v^{k+1} = 0, \\
v^{k+1} \cdot n = 0 \text{ on } \partial M'.
\end{cases}$$

(34)
Let $\Delta t = T/K$, $\mathbf{v}^k \approx \mathbf{v}(x, y, k\Delta t)$, and $\mathbf{v}^{k+\frac{1}{2}}$ represents an intermediate value between $\mathbf{v}^k$ and $\mathbf{v}^{k+1}$, etc.

**First Step:**

$$
\begin{cases}
\frac{\mathbf{v}^{k+\frac{1}{2}} - \mathbf{v}^k}{\Delta t} + \bar{U}_0 \mathbf{v}^{k+\frac{1}{2}}_x + f_k \times \mathbf{v}^k + \nabla \phi_0^k + G_0^k = 0, \\
\mathbf{v}^{k+\frac{1}{2}}|_{x=0} = 0,
\end{cases}
$$

Here

$$G_0^k = \begin{pmatrix}
\int_{-L}^0 B(u^k, v^k, w^k; u^k) \bar{U}_0(z) \, dz \\
\int_{-L}^0 B(u^k, v^k, w^k; v^k) \bar{U}_0(z) \, dz + f \bar{U}_0 \sqrt{L_3}
\end{pmatrix}.
$$

**Second Step:** (projection method)

$$
\begin{cases}
\frac{\mathbf{v}^{k+1} - \mathbf{v}^{k+\frac{1}{2}}}{\Delta t} + \nabla(\phi_0^{k+1} - \phi_0^k) = 0, \\
\nabla \cdot \mathbf{v}^{k+1} = 0, \\
\mathbf{v}^{k+1} \cdot \mathbf{n} = 0 \text{ on } \partial \mathcal{M}'.
\end{cases}
$$
Let $\Delta t = T/K$, $\mathbf{v}^k \approx \mathbf{v}(x, y, k\Delta t)$, and $\mathbf{v}^{k+\frac{1}{2}}$ represents an intermediate value between $\mathbf{v}^k$ and $\mathbf{v}^{k+1}$, etc.

First Step:

\[
\begin{cases}
\frac{\mathbf{v}^{k+\frac{1}{2}} - \mathbf{v}^k}{\Delta t} + \mathbf{U}_0 \mathbf{v}^{k+\frac{1}{2}}_x + f_k \times \mathbf{v}^k + \nabla \phi^k_0 + G^k_0 = 0, \\
\mathbf{v}^{k+\frac{1}{2}}|_{x=0} = 0, \\
\end{cases}
\]

Here

\[
G^k_0 = \begin{pmatrix}
\int_{-L_3}^{0} B(u^k, v^k, w^k; u^k) \mathcal{U}_0(z) \, dz \\
\int_{-L_3}^{0} B(u^k, v^k, w^k; v^k) \mathcal{U}_0(z) \, dz + f \mathbf{U}_0 \sqrt{L_3}
\end{pmatrix}.
\]

Second Step: (projection method)

\[
\begin{cases}
\frac{\mathbf{v}^{k+1} - \mathbf{v}^{k+\frac{1}{2}}}{\Delta t} + \nabla (\phi^{k+1}_0 - \phi^k_0) = 0, \\
\nabla \cdot \mathbf{v}^{k+1} = 0, \\
\mathbf{v}^{k+1} \cdot \mathbf{n} = 0 \text{ on } \partial M'.
\end{cases}
\]
From the Second Step, we can find $\phi_0^{k+1}$ by solving the Neumann problem

\[
\begin{align*}
\Delta \phi_0^{k+1} &= \Delta \phi_0^k + \frac{\nabla \cdot \mathbf{v}^{k+\frac{1}{2}}}{\Delta t}, \\
\nabla \phi_0^{k+1} \cdot \mathbf{n} &= \frac{\mathbf{v}^{k+\frac{1}{2}}}{\Delta t}, \quad \text{on } \partial \mathcal{M}',
\end{align*}
\]

(35)

and imposing the compatibility condition

\[
\int_{\mathcal{M}'} \phi_0^{k+1} \, dx \, dy = 0.
\]
Given the mesh size $h = (\Delta x, \Delta y)$, we have the following stability result:

**Lemma 1**

*If $\Delta t$ and $S(h)$ satisfy the conditions

\[
\Delta t \, S^4(h) \leq \frac{1}{c_1^2 K_4}, \quad \Delta t \leq \frac{1}{8}, \text{ where } S^2(h) = \frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2},
\]

then, for $0 \leq n \leq N_T$, we have

\[
|v^n_h|^2 \leq K_4, \quad (\Delta t)^3 \sum_{k=1}^{N_T} |\nabla_h \phi_h^k|^2 \leq K_4.
\]
The subcritical and supercritical modes

We rewrite (29) in the matrix form as follows:

\[
\frac{\partial U_n}{\partial t} + E_n \frac{\partial U_n}{\partial x} + F_n \frac{\partial U_n}{\partial y} + G_n = 0. \tag{36}
\]

Here,

\[
U_n = \begin{pmatrix} u_n \\ v_n \\ \psi_n \end{pmatrix}, \quad E_n = \begin{pmatrix} \bar{U}_0 & 0 & -1 \\ 0 & \bar{U}_0 & \frac{1}{\lambda_n} \\ \frac{-N^2}{\lambda_n} & 0 & \bar{U}_0 \end{pmatrix}, \quad F_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \frac{1}{\lambda_n} \\ 0 & \frac{-N^2}{\lambda_n} & 0 \end{pmatrix} \tag{37}
\]

and

\[
G_n = \begin{pmatrix} -fv_n + \int_{-L_3}^{0} B(u, v, w; u)\mathcal{U}_n(z)dz \\ fu_n + \int_{-L_3}^{0} B(u, v, w; v)\mathcal{U}_n(z)dz \\ \int_{-L_3}^{0} B(u, v, w; \psi)\mathcal{W}_n(z)dz \end{pmatrix}. \tag{38}
\]
Change variables:

\[
\begin{pmatrix}
\xi_n \\
v_n \\
\eta_n
\end{pmatrix}
= \begin{pmatrix}
u_n - \frac{\psi_n}{N} \\
v_n \\
u_n + \frac{\psi_n}{N}
\end{pmatrix},
\begin{pmatrix}
u_n \\
\alpha_n \\
\beta_n
\end{pmatrix}
= \begin{pmatrix}
u_n + \frac{\psi_n}{N} \\
v_n \\
v_n - \frac{\psi_n}{N}
\end{pmatrix}.
\]
Boundary conditions for the subcritical modes:

\[
\begin{align*}
\xi_n(0, y, t) &= 0, \\
v_n(0, y, t) &= 0, \\
\eta_n(L_1, y, t) &= 0.
\end{align*}
\]

\[
\begin{align*}
\alpha_n(x, L_2, t) &= 0, \\
\beta_n(x, 0, t) &= 0.
\end{align*}
\]

(40)

Figure 12: Boundary conditions for the subcritical modes
Boundary conditions for the supercritical modes:

\[
\begin{align*}
\xi_n(0, y, t) &= 0, \\
v_n(0, y, t) &= 0, \\
\eta_n(0, y, t) &= 0.
\end{align*}
\]

\[
\begin{align*}
\alpha_n(x, L_2, t) &= 0, \\
\beta_n(x, 0, t) &= 0.
\end{align*}
\]  

(41)

Figure 13: Boundary conditions for the supercritical modes
Remark 2

- The boundary conditions (40) and (41) are different from those proposed in RTT08.
- The well-posedness of the higher modes will be studied elsewhere.
Numerical Schemes for the subcritical modes

Splitting method:
The First Step: in $x$-direction

$$\frac{U_{n}^{k+\frac{1}{2}} - U_{n}^{k}}{\Delta t} + E_{n} \frac{\partial U_{n}^{k+\frac{1}{2}}}{\partial x} + G_{n}^{k} = 0. \quad (42)$$

The Second Step: in $y$-direction

$$\frac{U_{n}^{k+1} - U_{n}^{k+\frac{1}{2}}}{\Delta t} + F_{n} \frac{\partial U_{n}^{k+1}}{\partial y} = 0. \quad (43)$$

Remark 3

This is a partly implicit scheme.
Numerical scheme for the subcritical modes

The First Step:

\[
\begin{align*}
\frac{\xi_{n,i,j}^{k+\frac{1}{2}} - \xi_{n,i,j}^k}{\Delta t} + (\bar{U}_0 + \frac{N}{\lambda_n})\frac{\xi_{n,i,j}^{k+\frac{1}{2}} - \xi_{n,i-1,j}^k}{\Delta x} &= S_{n,i,j}^{k,1}, \quad i = 2, \ldots, I + 1, \\
\frac{v_{n,i,j}^{k+\frac{1}{2}} - v_{n,i,j}^k}{\Delta t} + \bar{U}_0\frac{v_{n,i,j}^{k+\frac{1}{2}} - v_{n,i-1,j}^k}{\Delta x} &= S_{n,i,j}^{k,2}, \quad i = 2, \ldots, I + 1, \\
\frac{\eta_{n,i,j}^{k+\frac{1}{2}} - \eta_{n,i,j}^k}{\Delta t} + (\bar{U}_0 - \frac{N}{\lambda_n})\frac{\eta_{n,i+1,j}^{k+\frac{1}{2}} - \eta_{n,i,j}^k}{\Delta x} &= S_{n,i,j}^{k,3}, \quad i = 1, \ldots, I, \\
\text{and } j = 1, \ldots, J + 1 \text{ in all cases,}
\end{align*}
\]

where

\[S_{n,i,j}^{k,1}, S_{n,i,j}^{k,2} \text{ and } S_{n,i,j}^{k,2} \text{ are nonlinear terms.}\]
Numerical scheme for the subcritical modes

The boundary conditions for $\xi_n^{k+\frac{1}{2}}$, $v_n^{k+\frac{1}{2}}$ and $\eta_n^{k+\frac{1}{2}}$ are

$$
\xi_{n,0,j}^{k+\frac{1}{2}} = 0, \quad v_{n,0,j}^{k+\frac{1}{2}} = 0, \quad \eta_{n,1,j}^{k+\frac{1}{2}} = 0, \quad \text{for } 0 \leq j \leq J, \quad (45)
$$

The Second Step

$$
\begin{align*}
\left\{ \begin{array}{l}
\frac{u_{n,i,j}^{k+1} - u_{n,i,j}^{k+\frac{1}{2}}}{\Delta t} = 0, \\
\frac{\alpha_{n,i,j}^{k+1} - \alpha_{n,i,j}^{k+\frac{1}{2}}}{\Delta t} - \frac{N}{\lambda_n} \frac{\alpha_{n,i,j+1}^{k+1} - \alpha_{n,i,j}^{k+1}}{\Delta y} = 0, \\
\frac{\beta_{n,i,j}^{k+1} - \beta_{n,i,j}^{k+\frac{1}{2}}}{\Delta t} + \frac{N}{\lambda_n} \frac{\beta_{n,i,j}^{k+1} - \beta_{n,i,j-1}^{k+1}}{\Delta y} = 0.
\end{array} \right.
\end{align*}
$$

The boundary conditions for $\alpha_n^{k+1}$, $\beta_n^{k+1}$ are

$$
\begin{align*}
\left\{ \begin{array}{l}
\alpha_{n,1,j}^{k+1} = 0, \quad \text{for } 0 \leq j \leq J, \\
\beta_{n,i,0}^{k+1} = 0, \quad \text{for } 0 \leq i \leq I.
\end{array} \right.
\end{align*}
$$

(47)
Numerical schemes for the supercritical modes

Splitting method:
The First Step:

\[
\begin{align*}
\frac{\xi^{k+\frac{1}{2}}_{n,i,j} - \xi^{k}_{n,i,j}}{\Delta t} + \left( \bar{U}_0 + \frac{N}{\lambda_n} \right) \frac{\xi^{k+\frac{1}{2}}_{n,i,j} - \xi^{k+\frac{1}{2}}_{n,i-1,j}}{\Delta x} &= S^{k,1}_{n,i,j}, i = 2, \ldots, I + 1, \\
\frac{v^{k+\frac{1}{2}}_{n,i,j} - v^{k}_{n,i,j}}{\Delta t} + \bar{U}_0 \frac{v^{k+\frac{1}{2}}_{n,i,j} - v^{k+\frac{1}{2}}_{n,i-1,j}}{\Delta x} &= S^{k,2}_{n,i,j}, i = 2, \ldots, I + 1, \\
\frac{\eta^{k+\frac{1}{2}}_{n,i,j} - \eta^{k}_{n,i,j}}{\Delta t} + \left( \bar{U}_0 - \frac{N}{\lambda_n} \right) \frac{\eta^{k+\frac{1}{2}}_{n,i,j} - \eta^{k+\frac{1}{2}}_{n,i-1,j}}{\Delta x} &= S^{k,3}_{n,i,j}, i = 1, \ldots, I, \\
\text{and } i = 1, \ldots, J + 1 \text{ in all cases.}
\end{align*}
\]

The boundary conditions for \( \xi^{k+\frac{1}{2}}_{n}, v^{k+\frac{1}{2}}_{n} \) and \( \eta^{k+\frac{1}{2}}_{n} \) are, for \( 0 \leq j \leq J \),

\[
\xi^{k+\frac{1}{2}}_{n,0,j} = 0, \quad v^{k+\frac{1}{2}}_{n,0,j} = 0, \quad \eta^{k+\frac{1}{2}}_{n,0,j} = 0.
\]
Numerical schemes for the supercritical modes

The second Step:

\[
\begin{align*}
\frac{u_{n,i,j}^{k+1} - u_{n,i,j}^{k+\frac{1}{2}}}{\Delta t} &= 0, \\
\frac{\alpha_{n,i,j}^{k+1} - \alpha_{n,i,j}^{k+\frac{1}{2}}}{\Delta t} - \frac{N \alpha_{n,i,j+1}^{k+1} - \alpha_{n,i,j}^{k+1}}{\lambda_n \Delta y} &= 0, \\
\frac{\beta_{n,i,j}^{k+1} - \beta_{n,i,j}^{k+\frac{1}{2}}}{\Delta t} + \frac{N \beta_{n,i,j}^{k+1} - \beta_{n,i,j-1}^{k+1}}{\lambda_n \Delta y} &= 0.
\end{align*}
\]

The boundary conditions for \(\alpha_n^{k+1}, \beta_n^{k+1}\) are

\[
\begin{align*}
\alpha_{n,i,j}^{k+1} &= 0, & \text{for } 0 \leq j \leq J, \\
\beta_{n,i,0}^{k+1} &= 0, & \text{for } 0 \leq i \leq I.
\end{align*}
\]
In our study, we only need to consider a small number of modes ($\leq 10$), and it is then appropriate to transform these integrals into the sums of the Fourier coefficients.
Consider two domains as follows:

The larger domain:
\[ \mathcal{M} = (0, L_1) \times (0, L_2) \times (-L_3, 0) \]

The middle-half domain:
\[ \mathcal{M}_1 = (L_1/4, 3L_1/4) \times (L_2/4, 3L_2/4) \times (-L_3, 0) \]

Figure 14: The larger domain $\mathcal{M}$ and the middle half domain $\mathcal{M}_1$. 
Strategies:

1. Given the initial and boundary conditions, we perform simulations on the larger domain.

2. We perform simulations on the middle-half domain using the initial and boundary conditions provided from Step 1.

3. We consider the data from Step 1 as the true solution, compare these two data from Step 1 and Step 2 in the middle-half domain and compute relative errors.
Numerical Experiments

In the simulation, the initial conditions are given by these scalar functions:

\[
\begin{align*}
  u(x, y, z, 0) &= \frac{x}{L_1} \frac{2\pi}{L_2} \sin \left( \frac{2\pi x}{L_1} \right) \cos \left( \frac{2\pi y}{L_2} \right) + \sin \left( \frac{4\pi x}{L_1} \right) \cos \left( \frac{4\pi y}{L_2} \right) \cos \left( \frac{\pi z}{H} \right), \\
  v(x, y, z, 0) &= - \frac{1}{L_1} \left( \sin \left( \frac{2\pi x}{L_1} \right) + \frac{2\pi x}{L_1} \cos \left( \frac{2\pi x}{L_1} \right) \right) \sin \left( \frac{2\pi y}{L_2} \right) \\
  &\quad + \frac{L_2}{L_1} \left( \sin^2 \left( \frac{4\pi x}{L_1} \right) + \sin \left( \frac{4\pi x}{L_1} \right) \sin \left( \frac{4\pi y}{L_2} \right) \cos \left( \frac{\pi z}{H} \right) \right) , \\
  w(x, y, z, 0) &= - \frac{4H}{L_1} \left( \sin \left( \frac{4\pi x}{L_1} \right) + \cos \left( \frac{4\pi x}{L_1} \right) \right) \cos \left( \frac{4\pi y}{L_2} \right) \sin \left( \frac{\pi z}{H} \right) , \\
  \phi(x, y, z, 0) &= \bar{U}_0 \sin \left( \frac{2\pi x}{L_1} \right) \sin \left( \frac{2\pi y}{L_2} \right) \left( \cos \left( \frac{\pi z}{H} \right) - \cos \left( \frac{2\pi z}{H} \right) \right), \\
  \psi(x, y, z, 0) &= \frac{\pi \bar{U}_0}{H} \sin \left( \frac{2\pi x}{L_1} \right) \sin \left( \frac{2\pi y}{L_2} \right) \left( 2 \sin \left( \frac{2\pi z}{H} \right) - \sin \left( \frac{\pi z}{H} \right) \right) .
\end{align*}
\] (52)
Boundary conditions: homogeneous conditions for the zero mode,

\[
\begin{align*}
    u_0(0, y, t) &= 0, & u_0(L_1, y, t) &= 0, \\
    v_0(0, y, t) &= 0, & v_0(x, 0, t) &= 0, & v_0(x, L_2, t) &= 0;
\end{align*}
\]

(53)

for the subcritical modes, i.e. when \( 1 \leq n < n_c \),

\[
\begin{align*}
    \xi_n(0, y, t) &= 0, & v_n(0, y, t) &= 0, & \eta_n(L_1, y, t) &= 0, \\
    \alpha_n(x, L_2, t) &= 0, & \beta_n(x, 0, t) &= 0;
\end{align*}
\]

(54)

and for the supercritical modes, i.e. when \( n > n_c \),

\[
\begin{align*}
    \xi_n(0, y, t) &= 0, & v_n(0, y, t) &= 0, & \eta_n(0, y, t) &= 0, \\
    \alpha_n(x, L_2, t) &= 0, & \beta_n(x, 0, t) &= 0.
\end{align*}
\]

(55)
The physical parameters:

- The length of the domain in $x$ direction: $L_1 = 10^3$ km,
- The length of the domain in $y$ direction: $L_2 = 500$ km,
- The length of the domain in $z$ direction: $L_3 = 10$ km,
- The constant reference velocity: $\bar{U}_0 = 20$ m/s,
- The Coriolis parameter: $f = 10^{-4}$,
- The Brunt–Väisälä (buoyancy) frequency: $N = 10^{-2}$,
- The final time: $T = 5 \times 10^4$ s,
- The number of modes: $N_{max} = 5$. 
The numerical parameters:

The number of time steps $N_T = 1600$, 
The number of mesh grids in $x$ $N_x = 400$, 
The number of mesh grids in $y$ $N_y = 200$, 
The number of mesh grids in $z$ $N_z = 40$. 
Figure 15: Top row: evolution of the solution $u$ in the $L^2$ and $L^\infty$ norms. Bottom row: evolution of the relative errors for $u$ in the $L^2$ and $L^\infty$ norms.
Figure 16: Top row: evolution of the solution v in the $L^2$ and $L^\infty$ norms. Bottom row: evolution of the relative errors for v in the $L^2$ and $L^\infty$ norms.
Figure 17: Top row: evolution of the solution $w$ in the $L^2$ and $L^\infty$ norms. Bottom row: evolution of the relative errors for $w$ in the $L^2$ and $L^\infty$ norms.
Figure 18: Top row: evolution of the solution $\psi$ in the $L^2$ and $L^\infty$ norms. Bottom row: evolution of the relative errors for $\psi$ in the $L^2$ and $L^\infty$ norms.
Figure 19: Top row: evolution of the solution $\phi$ in $L^2$ and $L^\infty$ norms. Bottom row: evolution of the relative errors for $\phi$ in $L^2$ and $L^\infty$ norms.
Figure 20: mean divergence
Summary

1. We propose suitable boundary conditions for one-layer or two-layer shallow water equations and numerical experiments are presented to confirm the suitability of the proposed boundary conditions.

2. We propose suitable boundary conditions for the 3D inviscid Primitive equations and numerical experiments are presented to confirm the suitability of the proposed boundary conditions.
Thank you !!