On the Asymptotic analysis and the existence of Time Periodic Solutions of the primitive equations

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joint work with Ming-Cheng Shiue

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Reynolds number $Vd/\nu < 5.71$ asymptotic stable.

1. the forcing term is time-periodic with period $T$, and
2. there exists a flow with Reynolds number less than 5.71 and this flow is equicontinuous in space variable for all time.
Reynolds number $Vd/\nu < 5.71$ asymptotically stable.

1. the forcing term is time-periodic with period $T$, and
2. there exists a flow with Reynolds number less than 5.71 and this flow is equicontinuous in space variable for all time.
The Primitive Equation

\[
\begin{align*}
    &\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + w \frac{\partial \mathbf{v}}{\partial z} + f \mathbf{v}^\perp + \nabla p = \nu_1 \Delta \mathbf{v} + \mu_1 \frac{\partial^2 \mathbf{v}}{\partial z^2} + F_1 \\
    &\frac{\partial p}{\partial z} = -\theta \\
    &\nabla \cdot \mathbf{v} + \frac{\partial w}{\partial z} = 0 \\
    &\frac{\partial \theta}{\partial t} + (\mathbf{v} \cdot \nabla) \theta + w \frac{\partial \theta}{\partial z} = \nu_2 \Delta \theta + \mu_2 \frac{\partial^2 \theta}{\partial z^2} + F_2
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Cylinderical domain

\[ \mathcal{M} = \mathcal{M}' \times (-h, 0), \]

Boundary Condition

\[ \Gamma_u : \frac{\partial v}{\partial z} = 0, \ w = 0, \ \frac{\partial \theta}{\partial z} + \alpha \theta = 0, \]

\[ \Gamma_b : \frac{\partial v}{\partial z} = 0, \ w = 0, \ \frac{\partial \theta}{\partial z} = 0, \]

\[ \Gamma_l : v \cdot \vec{n} = 0, \ \frac{\partial v}{\partial \vec{n}} \times \vec{n} = 0, \ \theta = 0. \]
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\( H = H_1 \times H_2, \quad V = V_1 \times V_2, \)

\[
H_1 = \left\{ \mathbf{v} \in (L^2(M))^2 : \int_{-h}^{0} \nabla \cdot \mathbf{v} \, dz = 0, \quad \mathbf{v} \cdot \vec{n} = 0, \text{ on } \Gamma_1 \right\},
\]

\[
V_1 = \left\{ \mathbf{v} \in (H^1(M))^2 : \int_{-h}^{0} \nabla \cdot \mathbf{v} \, dz = 0, \quad \mathbf{v} \cdot \vec{n} = 0, \text{ on } \Gamma_1 \right\},
\]

\[
H_2 = L^2(M), \quad V_2 = H^1(M).
\]
Cao-Titi, Kobelkov (2006)

Let $F_1 = 0$, $F_2 \in H^1(M)$, $(v_0, \theta_0) \in V_1 \times V_2$ and $T > 0$, then there exists a unique strong solution $(v, \theta)$ to the system of 3D viscous Primitive equations on the interval $[0, T]$, which depends on the initial data continuously in $H_1 \times H_2$. 
Medjo (2010)

\[ M_1 = \int_0^T \left| \frac{dF_2(t)}{dt} \right|^2_{L^2} dt. \]

\[ \sup |F(t)|_{L^2} \leq M \]

Set \( F_1(t) \equiv 0 \) and suppose \( F_2(t) = F_2(t + T) \). There exists a unique positive time periodic strong solution (with period \( T \)) of the primitive equation.
Reformulation of $w$ and $p$

\[ w(x, y, z, t) = - \int_{-h}^{z} \nabla \cdot \mathbf{v}(x, y, \xi, t) \, d\xi \]

\[ p(x, y, z, t) = p_0(x, y, t) - \int_{-h}^{z} \theta(x, y, \xi, t) \, d\xi \]
Integral differential equation

\[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \left( \int_{-h}^{z} \nabla \cdot \mathbf{v}(x, y, \xi, t) \, d\xi \right) \frac{\partial \mathbf{v}}{\partial z} + f \mathbf{v}^\perp \]

\[ + \nabla p_0 - \nabla \left( \int_{-h}^{z} \theta(x, y, \xi, t) \, d\xi \right) = \nu_1 \triangle \mathbf{v} + \mu_1 \frac{\partial^2 \mathbf{v}}{\partial z^2} + F_1, \]

\[ \frac{\partial \theta}{\partial t} + (\mathbf{v} \cdot \nabla) \theta - \left( \int_{-h}^{z} \nabla \cdot \mathbf{v}(x, y, \xi, t) \, d\xi \right) \frac{\partial \theta}{\partial z} \]

\[ = \nu_2 \triangle \theta + \mu_2 \frac{\partial^2 \theta}{\partial z^2} + F_2, \]

\[ \left. \frac{\partial \mathbf{v}}{\partial z} \right|_{\Gamma_u} = \left. \frac{\partial \mathbf{v}}{\partial z} \right|_{\Gamma_b} = 0, \quad \mathbf{v} \cdot \mathbf{n} \bigg|_{\Gamma_l} = 0, \quad \left. \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \times \mathbf{n} \right|_{\Gamma_l} = 0, \]

\[ \left. \frac{\partial \theta}{\partial z} \right|_{\Gamma_u} = \left. \frac{\partial \theta}{\partial z} \right|_{\Gamma_b} = 0, \quad \left. \frac{\partial \theta}{\partial \mathbf{n}} \right|_{\Gamma_l} = 0. \]
\[ \bar{v} = \frac{1}{h} \int_{-h}^{0} v(x, y, z, t) \, dz, \]
\[ \tilde{v} = v - \bar{v}. \]
\[
\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla)\bar{v} + (\tilde{v} \cdot \nabla)\tilde{v} + (\nabla \cdot \tilde{v})\tilde{v} + f\tilde{v}^\perp \\
+ \nabla p_0 - \int_{-h}^{z} \nabla \theta(x, y, \xi, t) \, d\xi = \nu_1 \Delta \bar{v} + \bar{F}_1,
\]
\[\nabla \cdot \bar{v} = 0,\]
\[\bar{v} \cdot \vec{n} |_{\partial \mathcal{M}'} = 0, \quad \frac{\partial \bar{v}}{\partial \vec{n}} \times \vec{n} |_{\partial \mathcal{M}'} = 0,\]
Perturbation

\[
\begin{align*}
\frac{\partial \tilde{v}}{\partial t} + (\tilde{v} \cdot \nabla)\tilde{v} - (\int_{-h}^{z} \nabla \cdot \tilde{v}(x, y, \xi, t) \, d\xi) \frac{\partial \tilde{v}}{\partial z} \\
&+ (\tilde{v} \cdot \nabla)\tilde{v} + (\bar{v} \cdot \nabla)\tilde{v} - (\tilde{v} \cdot \nabla)\tilde{v} + (\nabla \cdot \tilde{v})\tilde{v} + f \tilde{v} \perp \\
&- \int_{-h}^{z} \nabla \theta(x, y, \xi, t) \, d\xi + \int_{-h}^{z} \nabla \theta(x, y, \xi, t) \, d\xi \\
&= \nu_1 \Delta \tilde{v} + \mu_1 \frac{\partial^2 \tilde{v}}{\partial z^2} + \tilde{F}_1,
\end{align*}
\]

\[
\begin{align*}
\left. \frac{\partial \tilde{v}}{\partial z} \right|_{\Gamma_u} = \left. \frac{\partial \tilde{v}}{\partial z} \right|_{\Gamma_b} = 0, & \quad \tilde{v} \cdot \hat{n} \bigg|_{\partial \Gamma_l} = 0, \\
\left. \frac{\partial \tilde{v}}{\partial \hat{n} \times \hat{n}} \right|_{\partial \Gamma_l} = 0.
\end{align*}
\]
Theorem 1

Let $F \in L^\infty(0, T; L^2(M)^3)$, $\partial F/\partial z \in L^\infty(0, T; L^2(M)^3)$, $(v_0, \theta_0) \in V_1 \times V_2$, $\partial v_0/\partial z \in L^4(M)^2$, $\partial \theta_0/\partial z \in L^4(M)$, and $T > 0$. Then there exists a unique strong solution $(v, \theta)$ to the system of 3D viscous Primitive equations on the interval $[0, T]$ with $\partial v/\partial z \in L^2(0, T; L^4(M)^2)$ and $\partial \theta/\partial z \in L^2(0, T; L^4(M))$. 
Asymptotic stability

Suppose \( F = (F_1, F_2), \partial F/\partial z \in L^\infty(0, \infty; (L^2(\mathcal{M}))^3) \). There exists a positive number \( \tilde{\gamma}_2 \) such that if

\[
|F|_{L^\infty(0, \infty; (L^2(\mathcal{M}))^3)}^2 + \left| \frac{\partial F}{\partial z} \right|_{L^\infty(0, \infty; (L^2(\mathcal{M}))^3)}^2 \leq \tilde{\gamma}_2, \tag{1}
\]

then for any two strong solutions \( (\mathbf{v}_1(t), \theta_1(t)) \) and \( (\mathbf{v}_2(t), \theta_2(t)) \) of the primitive equations, we have

\[
\lim_{t \to \infty} \left( |\mathbf{v}_1(t) - \mathbf{v}_2(t)|_{L^2}^2 + |\theta_1(t) - \theta_2(t)|_{L^2}^2 \right) = 0. \tag{2}
\]

The convergence rate in (2) is exponential.
Let $F = (F_1, F_2) \in L^\infty(0, \infty; (L^2(M))^3)$ and
$\partial F/\partial z \in L^\infty(0, \infty; L^2(M)^3)$ be nontrivial and periodic in time with period $T$. Assume that $|F|_{L^\infty(0,\infty;(L^2(M))^3)}$ and $|\partial F/\partial z|_{L^\infty(0,\infty;(L^2(M))^3)}$ are small enough, then there exists a time periodic strong solution $(\nu, \theta)$ to the primitive equation. Moreover, any other strong solution tends to this time-periodic solution asymptotically in $L^2$ sense.
Lemma

Assumption:

\[ |\mathbf{v}_0|^2_{H^1} + |\theta_0|^2_{H^1} + |\frac{\partial \mathbf{v}_0}{\partial z}|^2_{L^4} + |\frac{\partial \theta_0}{\partial z}|^2_{L^4} \leq \gamma_1, \]
\[ |F|^2_{L^\infty(0,\infty;(L^2(M))^3)} + |\frac{\partial F}{\partial z}|^2_{L^\infty(0,\infty;(L^2(M))^2)} \leq \gamma_2, \]

then we have

\[
\sup_{t \geq 0} \left\{ |\mathbf{v}(t)|^2_{H^1} + |\theta(t)|^2_{H^1} + \left| \frac{\partial \mathbf{v}(t)}{\partial z} \right|^2_{L^4} + \left| \frac{\partial \theta(t)}{\partial z} \right|^2_{L^4} \right\} \leq C(\gamma_1, \gamma_2),
\]

(3)

\[ C(\gamma_1, \gamma_2) \downarrow 0 \text{ as } \gamma_1 + \gamma_2 \downarrow 0. \]
Sketch of Proof for the Main Theorem

Step 1. Let

\[ \delta = \frac{\min\{\nu_1, \mu_1, \nu_2, \mu_2, \mu_2 \alpha \}}{8(1 + 2h + 2h^2 + c^*_1)}. \]

Choose \( \gamma_1 \) and \( \gamma_2 \) small enough so that the constant \( C(\gamma_1, \gamma_2) \) in the Lemma satisfies

\[ C(\gamma_1, \gamma_2) < \frac{\delta^4}{4c^*_2 \left( \frac{h^2}{\nu_1} + \delta \right)^2}. \]

Let \((v'(x, y, z, t), \theta'(x, y, z, t))\) the solution with initial condition \((v_0, \theta_0)\), where \((v_0, \theta_0)\) satisfies the assumption in the Lemma. For another solution \((v''(x, y, z, t), \theta''(x, y, z, t))\), we define

\[ (\tilde{u}, \eta) = (v'', \theta'') - (v', \theta'). \]
The difference 
\((\tilde{u}, \eta)\) satisfies

\[
\frac{\partial \tilde{u}}{\partial t} + (\tilde{u} \cdot \nabla) \tilde{u} - \left( \int_{-h}^{z} (\nabla \cdot \tilde{u}) \, d\xi \right) \frac{\partial \tilde{u}}{\partial z} + f k \times \tilde{u} \\
+ (\tilde{u} \cdot \nabla) v' + (v' \cdot \nabla) \tilde{u} - \left( \int_{-h}^{z} (\nabla \cdot v') \, d\xi \right) \frac{\partial \tilde{u}}{\partial z} \\
- \left( \int_{-h}^{z} (\nabla \cdot \tilde{u}) \, d\xi \right) \frac{\partial v'}{\partial z} - \int_{-h}^{z} \nabla \eta \, d\xi = \nu_1 \Delta \tilde{u} + \mu_1 \frac{\partial^2 \tilde{u}}{\partial z^2}.
\]

\[
\frac{\partial \eta}{\partial t} + (\tilde{u} \cdot \nabla)(\theta' + \eta) - \left( \int_{-h}^{z} \nabla \cdot \tilde{u} \, d\xi \right) \frac{\partial (\theta' + \eta)}{\partial z} + (v' \cdot \nabla) \eta \\
- \left( \int_{-h}^{z} \nabla \cdot v' \, d\xi \right) \frac{\partial \eta}{\partial z} = \nu_2 \Delta \eta + \mu_2 \frac{\partial^2 \eta}{\partial z^2},
\]
\[ \int_{-h}^{0} \nabla \cdot \tilde{u} \, d\xi = 0, \]
\[ \frac{\partial \tilde{u}}{\partial z}|_{\Gamma_u} = \frac{\partial \tilde{u}}{\partial z}|_{\Gamma_b} = 0, \quad \tilde{u} \cdot \vec{n}|_{\Gamma_l} = 0, \quad \frac{\partial \tilde{u}}{\partial z} \times \vec{n}|_{\Gamma_l} = 0, \]
\[ \frac{\partial \eta}{\partial z}|_{\Gamma_u} = \frac{\partial \eta}{\partial z}|_{\Gamma_b} = 0, \quad \frac{\partial \eta}{\partial n}|_{\Gamma_l} = 0. \]
Taking the $L^2$ inner product of $(\tilde{u}, \eta)$ with the difference equation, we obtain

\[
\frac{1}{2} \frac{d}{dt} |\tilde{u}|^2_{L^2} + \nu_1 |\nabla \tilde{u}|^2_{L^2} + \mu_1 \left| \frac{\partial \tilde{u}}{\partial z} \right|^2_{L^2} \\
= -b(\tilde{u}, \nu', \tilde{u}) + \int_{M} \left( \int_{-h}^{z} \nabla \eta \, d\xi \right) \tilde{u} \, dM,
\]

\[
\frac{1}{2} \frac{d}{dt} |\eta|^2_{L^2} + \nu_2 |\nabla \eta|^2_{L^2} + \mu_2 |\eta_z|^2_{L^2} = -b(\tilde{u}, \theta', \eta).
\]
\[ |b(\tilde{u}, v', \tilde{u})| = |\int_M (\tilde{u} \cdot \nabla) v' \cdot \tilde{u} \, dM - \int_M (\int_{-h}^{\tilde{z}} \nabla \cdot \tilde{u} \, d\xi) \frac{\partial v'}{\partial z} \cdot \tilde{u} \, dM| \]

\[ \leq \int_M |\tilde{u}| |v'| |\nabla \tilde{u}| \, dM + \int_M |\nabla \cdot \tilde{u}|_{L^2} (z) |\frac{\partial v'}{\partial z}| |\tilde{u}| \, dM \]

\[ \leq |\nabla \tilde{u}|_{L^2} |v'|_{L^3} |\tilde{u}|_{L^6} + c |\nabla \tilde{u}|_{L^2} |\frac{\partial v'}{\partial z}|_{L^4} |\tilde{u}|_{L^4} \]

\[ \leq (\text{By Sobolev and Ladyzhenskaya's inequalities in } \mathbb{R}^3) \]

\[ \leq c |v'|_{L^2}^{1/2} |v'|_{H^1}^{1/2} |\nabla \tilde{u}|_{L^2} |\tilde{u}|_{H^1} + c |\nabla \tilde{u}|_{L^2} |\frac{\partial v'}{\partial z}|_{L^4} |\tilde{u}|_{L^4} \]

\[ \leq C(\gamma_1, \gamma_2) (|\nabla \tilde{u}|_{L^2}^2 + |\frac{\partial \tilde{u}}{\partial z}|_{L^2}^2) \]
\[ |b(\tilde{u}, \theta', \eta)| = \left| \int_{\mathcal{M}} \tilde{u} \cdot (\nabla \theta') \eta \, d\mathcal{M} - \int_{\mathcal{M}} (\int_{-h}^z \nabla \cdot \tilde{u} \, d\xi) \frac{\partial \theta'}{\partial z} \eta \, d\mathcal{M} \right| \]

\[ \leq \int_{\mathcal{M}} |\tilde{u}| \|\nabla \theta'| \| \eta\| \, d\mathcal{M} + c \int_{\mathcal{M}} \|\nabla \cdot \tilde{u}\|_{L^2(z)} \| \frac{\partial \theta'}{\partial z}\| \| \eta\| \, d\mathcal{M} \]

\[ \leq |\tilde{u}|_{L^3} \|\nabla \theta'|_{L^2} \| \eta\|_{L^6} + c |\nabla \tilde{u}|_{L^2} \| \frac{\partial \theta'}{\partial z}\|_{L^3} \| \eta\|_{L^6} \]

\[ \leq (\text{By Sobolev and Ladyzhenskaya's inequalities in } \mathbb{R}^3) \]

\[ \leq |\tilde{u}|^{1/2} \|\tilde{u}\|^{1/2}_{H^1} \|\nabla \theta'|_{L^2} \| \eta\|_{H^1} + c |\nabla \tilde{u}|_{L^2} \| \frac{\partial \theta'}{\partial z}\|_{L^4} \| \eta\|_{H^1} \]

\[ \leq C'(\gamma_1, \gamma_2) \left( |\nabla \tilde{u}|_{L^2}^2 + |\nabla \eta|_{L^2}^2 + |\frac{\partial \eta}{\partial z}|_{L^2}^2 \right) \]
\[
\left| \int_{\mathcal{M}} \left( \int_{-h}^{z} \nabla \eta \, d\xi \right) \tilde{u} \, dM \right| \leq h|\eta|_{L^2} |\nabla \tilde{u}|_{L^2} \leq \frac{\nu_1}{2} |\nabla \tilde{u}|_{L^2}^2 + c|\eta|_{L^2}^2
\]

where \( c \) depends on \( h \) and \( \nu_1 \).
\[ \frac{d}{dt} |\tilde{u}|_{L^2}^2 + \delta (|\nabla \tilde{u}|_{L^2}^2 + |\frac{\partial \tilde{u}}{\partial z}|_{L^2}^2) \leq c |\eta|_{L^2}^2, \]

\[ \frac{d}{dt} |\eta|_{L^2}^2 + 2\delta (|\nabla \eta|_{L^2}^2 + |\frac{\partial \eta}{\partial z}|_{L^2}^2) \leq C(\gamma_1, \gamma_2) |\nabla \tilde{u}|_{L^2}^2, \]
\[
|\tilde{u}(t)|_{L^2}^2 + \delta e^{-\delta t} \int_0^t e^{\delta s} (|\nabla \tilde{u}|_{L^2}^2 + |\frac{\partial \tilde{u}}{\partial z}|_{L^2}^2) \, ds \\
\leq e^{-\delta t} |u(0)|_{L^2}^2 + ce^{-\delta t} \int_0^t e^{\delta s} |\eta|_{L^2}^2 \, ds,
\]

\[
|\eta(t)|_{L^2}^2 \leq e^{-2\delta t} |\eta(0)|_{L^2}^2 + C(\gamma_1, \gamma_2)e^{-2\delta t} \int_0^t e^{2\delta s} |\nabla \tilde{u}|_{L^2}^2 \, ds.
\]

\[
|\tilde{u}|_{L^2}^2 + |\eta(t)|_{L^2}^2 \leq e^{-\delta t} |\tilde{u}(0)|_{L^2}^2 + \left( e^{-2\delta t} + \frac{ce^{-\delta_1 t}}{\delta} \right) |\eta(0)|_{L^2}^2. \tag{4}
\]
Now, set for $m > k$,

\[ v''(x, y, z, t) = v'(x, y, z, t + (m - k)T), \quad \theta''(x, y, z, t) = \theta'(x, y, z, t + (m - k)T) \]

By (4), we see that $(v'', \theta'')$ is a Cauchy sequence.
Put $t = kT$. We see that $(v''(mT), \theta''(mT))$ converges to some $(v''_0, \theta''_0)$.

We can show that the strong solution with initial condition $(v''_0, \theta''_0)$ is a time periodic solution and this solution is a global attractor of the solutions of the primitive equations.
Thank you very much for your attention!!