Introduction to Hamilton-Jacobi Equations and Periodic Homogenization

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August 22, 2012
A Hamilton-Jacobi equation is a first order equation

\[
\begin{align*}
H(Du, u, x) &= 0, \quad x \in \Omega \\
u &= 0, \quad x \in \partial \Omega
\end{align*}
\]

May not have $C^1$ solutions

\[
\begin{align*}
|u'| &= 1, \quad x \in (-1, 1) \\
u(-1) &= u(1) = 0
\end{align*}
\]

Solutions are defined in viscosity sense. [Crandall-Lions’83]
Viscosity Solution 1

- Given $u \in C(\Omega)$ and $x \in \Omega$.
  Define the super-differential and sub-differential of $u$ at $x$:

  $$D^+ u(x) = \{ p \in \mathbb{R}^n : u(y) \leq u(x) + p \cdot (y - x) + o(|y - x|), \ y \to x \}$$

  $$D^- u(x) = \{ p \in \mathbb{R}^n : u(y) \geq u(x) + p \cdot (y - x) + o(|y - x|), \ y \to x \}$$

- $u$ is a viscosity subsolution if

  $$H(p, u(x), x) \leq 0, \ \forall x \in \Omega, \ p \in D^+ u(x).$$

- $u$ is a viscosity supersolution if

  $$H(p, u(x), x) \geq 0, \ \forall x \in \Omega, \ p \in D^- u(x).$$

- $u \in C(\Omega)$ is a viscosity solution if $u$ is both a subsolution and supersolution.
Viscosity Solution 2

- $u \in USC(\Omega)$ is a viscosity subsolution if for any $\phi \in C^1(\Omega)$ such that $u - \phi$ reaches maximum at $x_0$ and $u(x_0) = \phi(x_0)$, then
  
  \[ H(D\phi(x_0), \phi(x_0), x_0) \leq 0. \]

- $u \in LSC(\Omega)$ is a viscosity supersolution if for any $\phi \in C^1(\Omega)$ such that $u - \phi$ reaches minimum at $x_0$ and $u(x_0) = \phi(x_0)$, then
  
  \[ H(D\phi(x_0), \phi(x_0), x_0) \geq 0. \]

- Viscosity solutions of 2nd order PDE [Crandall-Ishii-Lions’92]
Viscosity Solution 3

- (Vanishing Viscosity) If $u^\epsilon$ is the smooth solution of
  \[ H(Du^\epsilon, u^\epsilon, x) = \epsilon \Delta u^\epsilon \]
  and $u^\epsilon \to u$ locally uniformly as $\epsilon \to 0$, then $u$ is a viscosity solution.

- $u(x) = 1 - |x|$ is the unique viscosity solution of
  \[
  \begin{cases}
  |u'| = 1, & x \in (-1, 1) \\
  u(-1) = u(1) = 0
  \end{cases}
  \]

- (Regularity) Suppose that $H(p, u, x)$ is coercive in variable $p$:
  \[
  \lim_{|p| \to +\infty} H(p, u, x) = +\infty \quad \text{uniformly in } u, x
  \]
  then the viscosity solution is locally Lipschitz continuous.
Consider the ODE with control:

\[
\begin{align*}
    y'(s) &= f(y(s), \alpha(s)), \quad t < s < T \\
    y(t) &= x
\end{align*}
\]

\(x \in \mathbb{R}^n\): initial point at time \(t\). \(T\): terminal time.

\(f : \mathbb{R}^n \times A \to \mathbb{R}^n\) bounded and Lipschitz. \(A\): compact subset in \(\mathbb{R}^m\)

\(\alpha(\cdot) \in A\): set of the admissible control:

\[A = \{\alpha : [t, T] \to A | \alpha(\cdot) \text{ is measurable}\}\].

Control problem: find \(\alpha(\cdot)\) which optimizes the cost functional:

\[C_{x,t}[\alpha(\cdot)] = \int_t^T r(y(s), \alpha(s)) \, ds + g(y(T)),\]

\(y(\cdot)\) solves the ODE.

\(r : \mathbb{R}^n \times A \to \mathbb{R}\): running cost. \(g : \mathbb{R}^n \to \mathbb{R}\): terminal cost.
Define the value function as

\[ u(x, t) = \inf_{\alpha(\cdot) \in A} C_{x, t}[\alpha(\cdot)] \]

(Hamilton-Jacobi-Bellman equation) \( u \) is the unique viscosity solution of the terminal value problem:

\[
\begin{cases}
  u_t + \min_{a \in A} \{ f(x, a) \cdot Du + r(x, a) \} = 0 & \text{in } \mathbb{R}^n \times (0, T) \\
  u(x, T) = g(x) & \text{on } \mathbb{R}^n \times \{ t = T \}.
\end{cases}
\]
Choose $A = B_{s_L}(0) \subset \mathbb{R}^n$, $f(x, a) = -V(x) + a \in \mathbb{R}^n$, $r(x, a) \equiv 0$:

$$\min_{a \in B_{s_L}(0)} \{(-V(x) + a) \cdot p\} = -V(x) \cdot p - s_L |p|$$

The viscosity solution of

$$\begin{cases}
  u_t - V(x) \cdot Du - s_L |Du| = 0 & \text{in } \mathbb{R}^n \times (0, T) \\
  u(x, T) = g(x) & \text{on } \mathbb{R}^n \times \{t = T\}
\end{cases}$$

is given by

$$u(x, t) = \inf_{\alpha(\cdot) \in A} g(y(T)),$$

where the infimum is over all trajectories $y : [t, T] \to \mathbb{R}^n$ satisfying

$$\begin{cases}
  y'(s) = -V(y(s)) + \alpha(s), & t < s < T \\
  y(t) = x
\end{cases}$$

and the control $\|\alpha(s)\| \leq s_L$. 
For the media involving microscopic "self-repeating" environments, the process to extract the macroscopic "average out".

Heat conduction in composite material:

\[
\begin{array}{l}
-\nabla \cdot (A^\epsilon \nabla u^\epsilon) = f, \quad x \in \Omega \\
u^\epsilon = 0, \quad x \in \partial \Omega
\end{array}
\]

\(A^\epsilon = A(x/\epsilon) \in \mathbb{R}^{n \times n}\): thermal conductivity tensor
\(A(y)\): 1-periodic, uniformly positive definite

Homogenization of elliptic PDE: as \(\epsilon \to 0\), \(u^\epsilon \to \bar{u}\) (in some sense) the solution of

\[
\begin{array}{l}
-\nabla \cdot (\bar{A} \nabla \bar{u}) = f, \quad x \in \Omega \\
\bar{u} = 0, \quad x \in \partial \Omega
\end{array}
\]

\(\bar{A}\): homogenized conductivity tensor
1D case

- 1D problem:

\[
\begin{cases}
- \frac{d}{dx} \left( a \left( \frac{x}{\epsilon} \right) \frac{du^\epsilon}{dx} \right) = f, \quad 0 < x < L \\
u^\epsilon(0) = u^\epsilon(L) = 0
\end{cases}
\]

\(a(y): 1\text{-periodic}, \quad 0 < \alpha \leq a(y) \leq \beta < \infty.\)

- As \(\epsilon \to 0\), \(u^\epsilon \to \bar{u}\) the solution of

\[
\begin{cases}
- \frac{d}{dx} \left( \bar{a} \frac{d\bar{u}}{dx} \right) = f, \quad 0 < x < L \\
\bar{u}(0) = \bar{u}(L) = 0
\end{cases}
\]

where \(\bar{a}\) is the harmonic mean of \(a(y)\):

\[
\bar{a} = \left( \int_0^1 \frac{1}{a(y)} dy \right)^{-1}
\]
Homogenization of H-J

- Periodic homogenization of Hamilton-Jacobi equation
  [Lions-Papanicolaou-Varadhan’86]:

  \[ u_\epsilon t + H \left( Du_\epsilon, \frac{x}{\epsilon} \right) = 0 \]

  As \( \epsilon \to 0 \), \( u_\epsilon \to \bar{u} \) the solution of

  \[ \bar{u}_t + \bar{H}(D\bar{u}) = 0 \]

  \( \bar{H} \): effective Hamiltonian

  - (i) \( H(p, y) \) is continuous and periodic in \( y \)
  - (ii) \( H(p, y) \) is bounded in \( y \) for bounded \( p \)
  - (iii) \( H(p, y) \) is coercive in \( p \) uniformly in \( y \):

  \[ |H(p, y)| \to \infty \text{ as } |p| \to \infty \]
Asymptotic Expansion

- Two-scale asymptotic expansion:

\[ u^\varepsilon(x, t) = u^0(x, t) + \varepsilon u^1 \left( x, \frac{x}{\varepsilon}, t \right) + \cdots \]

\( x \): slow variable, \( y = \frac{x}{\varepsilon} \): fast variable
\( u^1(x, y, t) \): periodic in \( y \)

\[ u^\varepsilon_t = u^0_t + \varepsilon u^1_t + \cdots \]

\[ Du^\varepsilon = D_x u^0 + D_y u^1 + \varepsilon D_x u^0 + \cdots \]

- Leading order:

\[ u^0_t + H(D_x u^0 + D_y u^1, y) = 0 \]

should be independent of variable \( y \)
Cell Problem

- (Cell Problem) Given any $P \in \mathbb{R}^n$, find unique number $\bar{H} = \bar{H}(P)$ such that the equation

$$H(P + Dv, y) = \bar{H}, \quad y \in \mathbb{T}^n$$

has a periodic solution $v(y)$.

- For $\lambda > 0$, let $v^{(\lambda)}$ be the unique periodic viscosity solution of

$$H(P + Dv^{(\lambda)}, y) = \lambda v^{(\lambda)}, \quad y \in \mathbb{T}^n$$

Due to the coercivity, $u^{(\lambda)}$ is Lipschitz continuous uniformly in $\lambda$. As $\lambda \to 0$, $\lambda u^{(\lambda)} \to \bar{H}$ uniformly in $y$ and $u^{(\lambda)} \to u$ the viscosity solution of the cell problem.
For $\epsilon > 0$, assume that $u^\epsilon$ is the unique viscosity solution of

\[
\begin{cases}
  u^\epsilon_t + H \left( Du^\epsilon, \frac{x}{\epsilon} \right) = 0 \\
  u^\epsilon(x, 0) = g(x)
\end{cases}
\]

Then as $\epsilon \to 0$, $u^\epsilon$ converges uniformly to $\bar{u}$ the unique viscosity solution of the following effective equation:

\[
\begin{cases}
  \bar{u}_t + \bar{H}(D\bar{u}) = 0 \\
  \bar{u}(x, 0) = g(x)
\end{cases}
\]

where $\bar{H}$ is given by the cell problem.

Proof: perturbed test function method [Evans’89]

Homogenization of nonlinear 2nd order PDE [Evans’92]
So solve the cell problem numerically, consider the evolution equation [Qian'03]:

\[
\begin{aligned}
&v_t + H(P + Dv, y) = 0 \quad \text{in } \mathbb{T}^n \times (0, \infty) \\
v(x, 0) = 0 \quad \text{on } \mathbb{T}^n \times \{t = 0\}
\end{aligned}
\]

Due to the coercivity of \( H \), the effective Hamiltonian can be approximated as:

\[
\bar{H} = - \lim_{t \to +\infty} \frac{v(x, t)}{t},
\]

which converges uniformly in \( x \).