Fixed Point Theorems for
Single-Valued and Set-Valued Mappings
on Complete Metric Spaces

Chih-Sheng Chuang, Lai-Jiu Lin*, and Wataru Takahashi
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we obtain some generalized fixed point theorems for single-valued and set-valued mappings on complete metric spaces.

Using these results, we give new and well-known fixed point theorems which are not proved by Caristi’s fixed point theorem directly.
1. Introduction

Let \((X, d)\) be a metric space.

A mapping \(T : X \rightarrow X\) is said to be **contractive** if there exists 
\(r \in [0, 1)\) such that

\[
d(Tx, Ty) \leq r \ d(x, y)
\]

for all \(x, y \in X\). Such a mapping is also called **\(r\)-contractive**.
1. Introduction

Let $(X, d)$ be a metric space. A mapping $T : X \rightarrow X$ is said to be contractive if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq r \, d(x, y)$$

for all $x, y \in X$. Such a mapping is also called $r$-contractive.

A mapping $T : X \rightarrow X$ is said to be Kannan [7] if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha(d(x, Tx) + d(y, Ty))$$

for all $x, y \in X$. 
A mapping $T : X \to X$ is said to be **contractively nonspreading** if there exists $\beta \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \beta(d(x, Ty) + d(y, Tx))$$

for all $x, y \in X$. 

**Notation (Chatterjea, 1972)**
A mapping \( T : X \rightarrow X \) is called **contractively hybrid** \([4]\) if there exists \( \gamma \in [0, \frac{1}{3}) \) such that

\[
d(Tx, Ty) \leq r \{d(Tx, y) + d(Ty, x) + d(x, y)\}
\]

for all \( x, y \in X \).
Theorem 1.1 (Zamfirescu, 1972)

Assume that:

$(X, d)$ is a complete metric space;

$T : X \rightarrow X$ is a mapping which satisfies one of the following:

(i) $T$ is contractive;

(ii) $T$ is Kannan;

(iii) $T$ is contractively nonspreading.

Then $T$ has a unique fixed point in $X$. 
Recent Work

(Hasegawa, Komiya, and Takahashi, SCIJP, 2011)

Recently, Hasegawa, Komiya, and Takahashi introduced the concept of **contractively generalized hybrid mappings** on metric spaces, and studied the fixed point theorems for such mappings on complete metric spaces.
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Recently, Hasegawa, Komiya, and Takahashi introduced the concept of **contractively generalized hybrid mappings** on metric spaces, and studied the fixed point theorems for such mappings on complete metric spaces.

Let $X$ be a metric space. A mapping $T : X \to X$ is called **contractively generalized hybrid** if there exist $\alpha, \beta \in \mathbb{R}$ and $r \in [0, 1)$ such that

$$\alpha d(Tx, Ty) + (1 - \alpha)d(x, Ty) \leq r\{\beta d(Tx, y) + (1 - \beta)d(x, y)\}$$

for all $x, y \in X$. 


For example,

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1. An \((\alpha, \beta, r)\)-contractively generalized hybrid mapping is \(r\)-contractive for \(\alpha = 1\) and \(\beta = 0\);

2. An \((\alpha, \beta, r)\)-contractively generalized hybrid mapping is contractively nonspreading for \(\alpha = 1 + r\) and \(\beta = 1\);

3. An \((\alpha, \beta, r)\)-contractively generalized hybrid mapping is contractively hybrid for \(\alpha = 1 + \frac{r}{2}\) and \(\beta = \frac{1}{2}\).
2. Preliminaries

Throughout this paper,
\( \mathbb{N} \): the set of positive integers;
\( \mathbb{R} \): the set of real numbers;
Banach limit

1. Let $\ell^\infty$ be the Banach space of bounded sequences with the supremum norm.

2. Let $\mu$ be an element of $(\ell^\infty)^*$ (the dual space of $\ell^\infty$).
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2. Let $\mu$ be an element of $(\ell^\infty)^*$ (the dual space of $\ell^\infty$).
3. Then we denote by $\mu(f)$ the value of $\mu$ at $f = (x_1, x_2, x_3, \ldots) \in \ell^\infty$.
4. Sometimes, we denote by $\mu_n x_n$ the value $\mu(f)$.
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4. Sometimes, we denote by $\mu_n x_n$ the value $\mu(f)$.
5. A linear functional $\mu$ on $\ell^\infty$ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. 
**Banach limit**

1. Let $\ell^\infty$ be the Banach space of bounded sequences with the supremum norm.

2. Let $\mu$ be an element of $(\ell^\infty)^*$ (the dual space of $\ell^\infty$).

3. Then we denote by $\mu(f)$ the value of $\mu$ at $f = (x_1, x_2, x_3, ...) \in \ell^\infty$.

4. Sometimes, we denote by $\mu_n x_n$ the value $\mu(f)$.

5. A linear functional $\mu$ on $\ell^\infty$ is called a mean if $\mu(e) = ||\mu|| = 1$, where $e = (1, 1, 1, ....)$.

6. For $x = (x_1, x_2, x_3, ....)$, a Banach limit on $\ell^\infty$ is an invariant mean, that is, $\mu_n x_n = \mu_n x_{n+1}$ for any $n \in \mathbb{N}$. 

7. If \( \mu \) is a Banach limit on \( \ell^\infty \), then for \( f = (x_1, x_2, x_3, \ldots) \in \ell^\infty \),

\[
\liminf_{n \to \infty} x_n \leq \mu_n x_n \leq \limsup_{n \to \infty} x_n.
\]

In particular, if \( f = (x_1, x_2, x_3, \ldots) \in \ell^\infty \) and \( x_n \to a \in \mathbb{R} \), then we have \( \mu(f) = \mu_n x_n = a \).
Lemma 2.1 (Hasegawa, Komiya and Takahashi, in press)

Let \((X, d)\) be a metric space, let \(\{x_n\}\) be a bounded sequence in \(X\) and let \(\mu\) be a mean on \(\ell^\infty\). If \(g : X \to \mathbb{R}\) is defined by

\[
g(y) = \mu_n d(x_n, y), \quad \forall y \in X,
\]

then \(g\) is a continuous function on \(X\).
Notations

Let \((X, d)\) be a metric space and let \(f\) be a function of \(X\) into \((-\infty, \infty] = \mathbb{R} \cup \{\infty\}\).

Then

1. \(f\) is proper if there exists \(x \in X\) such that \(f(x) < \infty\). \(f\) is lower semicontinuous if for any \(r \in \mathbb{R}\), the set \(\{x \in X : f(x) \leq r\}\) is closed.

2. \(f\) is bounded below if there exists \(M \in \mathbb{R}\) such that

\[
M \leq f(x), \quad \forall x \in X.
\]
Caristi’s fixed point theorem

Let \((X, d)\) be a complete metric space and let \(\psi : X \rightarrow (\infty, \infty]\) be a proper, bounded below, and lower semicontinuous function. Let \(T : X \rightarrow X\) be a mapping such that for each \(x \in X\),

\[
d(x, Tx) + \psi(Tx) \leq \psi(x).
\]

Then there exists \(\bar{x} \in X\) such that \(T\bar{x} = \bar{x}\).
Notations.

Let \((X, d)\) be a metric space and let \(P(X)\) be the class of all nonempty subsets of \(X\).

A mapping of \(X\) into \(P(X)\) is called a *set-valued mapping*, or a *multi-valued mapping*.

For a set-valued mapping \(T : X \to P(X)\), we denote by \(F(T)\) and \(SF(T)\) the fixed point set for \(T\) and the strict fixed point set for \(T\) respectively, i.e.,

\[
F(T) = \{z \in X : z \in Tz\}, \quad SF(T) = \{z \in X : \{z\} = Tz\}.
\]
Notations.

For $x \in X$ and $A \subset X$, define

$$d(x, A) = \inf \{d(x, y) : y \in A\}.$$  

Let $BC(X)$ be the class of all nonempty bounded closed subsets of $X$.
For $A, B \in BC(X)$, define

$$\delta(A, B) = \sup \{d(x, B) : x \in A\}.$$
Theorem 2.2.

Let \((X, d)\) be a metric space and let \(BC(X)\) be the class of all nonempty bounded closed subsets of \(X\). For \(A, B \in BC(X)\), define

\[
H(A, B) = \max\{\delta(A, B), \delta(B, A)\}.
\]

Then, \(H\) is a metric on \(BC(X)\).
Notations.

Let $T$ be a mapping of $X$ into $BC(X)$. Then $T$ is called \textit{nonexpansive} if

$$H(Tx, Ty) \leq d(x, y), \quad x, y \in X.$$
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\[ H(Tx, Ty) \leq d(x, y), \quad x, y \in X. \]
If there exists $\alpha \in \mathbb{R}$ with $0 \leq \alpha < 1$ such that
\[ H(Tx, Ty) \leq \alpha d(x, y), \quad x, y \in X, \]
then $T$ is called an $\alpha$-\textit{contraction}.
Notations.

Let $T$ be a mapping of $X$ into $BC(X)$. Then $T$ is called nonexpansive if

$$H(Tx, Ty) \leq d(x, y), \quad x, y \in X.$$ 

If there exists $\alpha \in \mathbb{R}$ with $0 \leq \alpha < 1$ such that

$$H(Tx, Ty) \leq \alpha d(x, y), \quad x, y \in X,$$

then $T$ is called an $\alpha$-contraction.

If $T$ is nonexpansive, then the real valued function $g$ on $X$ defined by

$$g(x) = d(x, Tx), \quad \forall x \in X$$

is continuous.
Theorem 2.3 (Nadler, 1969).

Let \((X, d)\) be a complete metric space and let \(T\) be an \(\alpha\)-contraction from \(X\) into \(BC(X)\).
Then, \(T\) has a fixed point \(z\) in \(X\), i.e., \(z \in Tz\).
3. Fixed Point Theorems for Single-Valued Mappings
Theorem 3.1.

Let \((X, d)\) be a complete metric space, let \(\mu\) be a mean on \(\ell^\infty\), let \(\{x_n\}\) be a bounded sequence in \(X\), and let \(\psi : X \to (-\infty, \infty]\) be a proper, bounded below, and lower semicontinuous function. Let \(T : X \to X\) be a mapping. Suppose that there exists \(m \in \mathbb{N} \cup \{0\}\) such that

\[
\mu_n d(x_n, T^m y) + \psi(Ty) \leq \psi(y), \quad \forall y \in X.
\]

Then there exists \(\bar{x} \in X\) such that

(a) \(\mu_n d(x_n, \bar{x}) = 0\);
(b) \(\bar{x} = \lim_{m \to \infty} T^m y\) for all \(y \in X\) with \(\psi(y) < \infty\);
(c) \(\psi(\bar{x}) = \inf_{u \in X} \psi(u)\);
(d) \(\bar{x}\) is a unique fixed point of \(T\).
Proof of Theorem 3.1.

Let $g : X \to [0, \infty)$ be a function defined by $g(z) := \mu_n d(x_n, z)$ for each $z \in X$. Take $y \in X$ with $\psi(y) < \infty$. Then we have:

$$\mu_n d(x_n, T^m T^k y) \leq \psi(T^k y) - \psi(T^{k+1} y)$$

for all $k \in \mathbb{N} \cup \{0\}$. Then

$$(*)_1 \; \{\psi(T^k y)\}_{k=0}^{\infty} \text{ is a decreasing sequence which is bounded below.}$$

$$(*)_2 \lim_{k \to \infty} \psi(T^k y) \text{ exists.}$$

Put $s = \lim_{k \to \infty} \psi(T^k y)$. Since

$$\sum_{k=0}^{N} g(T^{m+k} y) = \sum_{k=0}^{N} \mu_n d(x_n, T^{m+k} y) \leq \psi(y) - \psi(T^{N+1} y)$$

for any $N \in \mathbb{N}$,
Proof of Theorem 3.1-cont.

we have that
\[ \sum_{k=0}^{\infty} g(T^{m+k}y) = \sum_{k=0}^{\infty} \mu_n d(x_n, T^{m+k}y) \leq \psi(y) - s < \infty. \]

Thus we have that
\[ \lim_{k \to \infty} g(T^k T^m y) = \lim_{k \to \infty} \mu_n d(x_n, T^k T^m y) = 0. \]

Hence for each \( \varepsilon > 0 \), there exists \( K \in \mathbb{N} \) such that for any \( k \geq K \),
\[ \mu_n d(x_n, T^k T^m y) < \frac{\varepsilon}{2}. \]

This implies that for all \( k_1, k_2 \geq K \),
\[ d(T^{k_1} T^m y, T^{k_2} T^m y) \leq \mu_n d(x_n, T^{k_1} T^m y) + \mu_n d(x_n, T^{k_2} T^m y) < \varepsilon. \]
Proof of Theorem 3.1-cont.

Then \( \{ T^k T^m y \} \) is a Cauchy sequence in \( X \), and there exists \( \bar{x} \in X \) such that \( \lim_{k \to \infty} T^k T^m y = \bar{x} \).
Proof of Theorem 3.1-cont.

Then \( \{ T^k T^m y \} \) is a Cauchy sequence in \( X \), and there exists \( \bar{x} \in X \) such that \( \lim_{k \to \infty} T^k T^m y = \bar{x} \).

By Lemma 2.1, we have that

\[
g(\bar{x}) = \lim_{k \to \infty} g(T^k T^m y) = 0.
\]

This implies that \( \mu_n d(x_n, \bar{x}) = 0 \).
Proof of Theorem 3.1-cont.

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This implies that \( \mu_n d(x_n, \bar{x}) = 0 \).

Thus we have that for any \( u \in X \) with \( \psi(u) < \infty \), there exists \( \bar{u} \in X \) such that \( \lim_{k \to \infty} T^k T^m u = \bar{u} \) and \( g(\bar{u}) = 0 \). This implies \( \mu_n d(x_n, \bar{u}) = 0 \).
Proof of Theorem 3.1-cont.

Then \( \{ T^k T^m y \} \) is a Cauchy sequence in \( X \), and there exists \( \bar{x} \in X \) such that \( \lim_{k \to \infty} T^k T^m y = \bar{x} \).

By Lemma 2.1, we have that

\[
g(\bar{x}) = \lim_{k \to \infty} g(T^k T^m y) = 0.
\]

This implies that \( \mu_n d(x_n, \bar{x}) = 0. \)

Thus we have that for any \( u \in X \) with \( \psi(u) < \infty \), there exists \( \bar{u} \in X \) such that \( \lim_{k \to \infty} T^k T^m u = \bar{u} \) and \( g(\bar{u}) = 0. \) This implies \( \mu_n d(x_n, \bar{u}) = 0. \) Since

\[
d(\bar{x}, \bar{u}) \leq d(\bar{x}, x_n) + d(x_n, \bar{u}),
\]

we have that

\[
d(\bar{x}, \bar{u}) \leq \mu_n d(\bar{x}, x_n) + \mu_n d(x_n, \bar{u}) = 0 + 0 = 0
\]
Proof of Theorem 3.1-cont.

Thus we have $\bar{u} = \bar{x}$. Therefore we have that $\bar{x} = \lim_{m \to \infty} T^k T^m z$ for all $z \in X$ with $\psi(z) < \infty$. 
Proof of Theorem 3.1-cont.

Thus we have $\bar{u} = \bar{x}$. Therefore we have that $\bar{x} = \lim_{m \to \infty} T^k T^m z$
for all $z \in X$ with $\psi(z) < \infty$.

By assumption, we have that

$$\psi(T\bar{x}) = \psi(T\bar{u}) \leq \mu_n d(x_n, T^m \bar{x}) + \psi(T\bar{x}) \leq \psi(\bar{x}).$$

Since $\psi$ is lower semicontinuous and $\lim_{k \to \infty} T^k T^m u = \bar{x}$ for all $u \in X$ with $\psi(u) < \infty$, we have that

$$\psi(\bar{x}) = \inf_{y \in X} \psi(y) \leq \psi(T\bar{x}) \leq \psi(\bar{x}),$$

and hence $\psi(\bar{x}) = \inf_{y \in X} \psi(y)$. 
Proof of Theorem 3.1-cont.

Since

\[ 0 \leq g(T^m\bar{x}) = \mu_n d(x_n, T^m\bar{x}) \leq \psi(\bar{x}) - \psi(T\bar{x}) \leq 0, \]

we have that \( g(T^m\bar{x}) = \mu_n d(x_n, T^m\bar{x}) = 0 \) and hence \( T^m\bar{x} = \bar{x} \).

Then we have that

\[ 0 \leq g(T\bar{x}) = \mu_n d(x_n, T\bar{x}) = \mu_n d(x_n, T^{m+1}\bar{x}) \leq \psi(\bar{x}) - \psi(T^2\bar{x}) \leq 0 \]

and hence \( g(T\bar{x}) = \mu_n d(x_n, T\bar{x}) = 0 \). Thus \( g(\bar{x}) = g(T\bar{x}) = 0 \) and hence \( T\bar{x} = \bar{x} \).
Proof of Theorem 3.1-cont.

Since

\[ 0 \leq g(T^m\bar{x}) = \mu_n d(x_n, T^m\bar{x}) \leq \psi(\bar{x}) - \psi(T\bar{x}) \leq 0, \]

we have that \( g(T^m\bar{x}) = \mu_n d(x_n, T^m\bar{x}) = 0 \) and hence \( T^m\bar{x} = \bar{x} \).

Then we have that

\[ 0 \leq g(T\bar{x}) = \mu_n d(x_n, T\bar{x}) = \mu_n d(x_n, T^{m+1}\bar{x}) \leq \psi(\bar{x}) - \psi(T^2\bar{x}) \leq 0 \]

and hence \( g(T\bar{x}) = \mu_n d(x_n, T\bar{x}) = 0 \). Thus \( g(\bar{x}) = g(T\bar{x}) = 0 \)

and hence \( T\bar{x} = \bar{x} \). We show that \( \bar{x} \) is a unique fixed point of \( T \).

Indeed, if \( v \) is a fixed point of \( T \), then

\[ 0 \leq g(v) = g(T^m v) \leq \psi(v) - \psi(Tv) = 0. \]

Hence we have \( v = \bar{x} \). Therefore \( \bar{x} \) is a unique fixed point of \( T \).
Corollary 3.1.

Assume that:

(a) \((X, d)\) is a complete metric space;
(b) \(\mu\) is a mean on \(\ell^\infty\);
(c) \(\{x_n\}\) is a bounded sequence in \(X\).
(d) \(\psi : X \to (-\infty, \infty)\) is a bounded below and lower semicontinuous function;
(e) \(T : X \to X\) is a mapping such that

\[
\mu_n d(x_n, Ty) + \psi(Ty) \leq \psi(y), \quad \forall y \in X.
\]

Then the following hold:

(i) \(T\) has a unique fixed point \(u\) in \(X\);
(ii) for every \(z \in X\), the sequence \(\{T^nz\}\) converges to \(u\).
Corollary 3.2.

Assume that:

(a) \((X, d)\) is a complete metric space;
(b) \(\mu\) is a mean on \(\ell^\infty\);
(c) \(\{x_n\}\) is a bounded sequence in \(X\).
(d) Let \(T : X \to X\) be a nonexpansive mapping. Suppose that there exists \(m \in \mathbb{N} \cup \{0\}\) such that

\[\mu_n d(x_n, T^m y) + d(Ty, T^2 y) \leq d(y, Ty), \quad \forall y \in X.\]

Then the following hold:

(i) \(T\) has a unique fixed point \(u\) in \(X\);
(ii) for every \(z \in X\), the sequence \(\{T^n z\}\) converges to \(u\).
Theorem 3.2 (Hasegawa, Komiya and Takahashi, in press)

Assume that:

(a) \((X, d)\) is a complete metric space;
(b) Let \(T\) be a mapping of \(X\) into itself. Suppose that there exist a real number \(r\) with \(0 \leq r < 1\) and an element \(x \in X\) such that \(\{T^n x\}\) is bounded and

\[
\mu_n d(T^n x, Ty) \leq r \mu_n d(T^n x, y), \quad \forall y \in X
\]

for some mean \(\mu\) on \(\ell^\infty\).

Then the following hold:

(i) \(T\) has a unique fixed point \(u \in X\);
(ii) for every \(z \in X\), the sequence \(\{T^n z\}\) converges to \(u\).
Proof of Theorem 3.2.

Let $r$ be a real number $r$ with $0 \leq r < 1$ and take $x \in X$ such that $\{T^n x\}$ is bounded. Define a function $\psi : X \to [0, \infty)$ as follows:

$$
\psi(y) = \frac{1}{1 - r} \mu_n d(T^n x, y), \quad \forall y \in X.
$$

Then we know from Lemma 2.1 that $\psi$ is continuous on $X$.

Furthermore we have that the following (1) and (2) are equivalent:

(1) $\mu_n d(T^n x, y) + \psi(Ty) \leq \psi(y), \quad \forall y \in X$;

(2) $\mu_n d(T^n x, Ty) \leq r \mu_n d(T^n x, y), \quad \forall y \in X$.

Thus from Theorem 3.1, we obtain the desired result.
Theorem 3.3 (Hasegawa, Komiya and Takahashi, in press).

Assume that:

(a) $(X, d)$ is a complete metric space;
(b) $T : X \rightarrow X$ is an $(\alpha, \beta, r)$-contractively generalized hybrid mapping such that

$$\beta \geq 0, \quad \alpha - r\beta > 0, \quad \text{and} \quad r < \frac{\alpha}{1 + \beta}.$$ 

Then the following hold:

(i) $T$ has a unique fixed point $u$ in $X$;
(ii) for every $z \in X$, the sequence $\{T^n z\}$ converges to $u$. 
Proof of Theorem 3.3.

For an \((\alpha, \beta, r)\)-contractively generalized hybrid mapping 

\[ T : X \to X \]

such that

\[ \beta \geq 0, \ \alpha - r\beta > 0, \ \text{and} \ r < \frac{\alpha}{1 + \beta}, \]

we know from [4] that the sequence \( \{T^n x\} \) for every \( x \in X \) is bounded. Fix \( x \in X \). Since \( T \) is a \((\alpha, \beta, r)\)-contractively generalized hybrid mapping, we have that for any \( y \in X \) and \( n \in \mathbb{N} \),

\[
\alpha d(T^{n+1}x, Ty) + (1 - \alpha)d(T^nx, Ty) \\
\leq r\{\beta d(T^{n+1}x, y) + (1 - \beta)d(T^nx, y)\}.
\]

Since \( \{T^nx\} \) is bounded, we can apply a Banach limit \( \mu \) to both sides of the inequality.
Proof of Theorem 3.3-cont.

Then we have that

\[ \alpha \mu_n d(T^{n+1}x, Ty) + (1 - \alpha) \mu_n d(T^n x, Ty) \]

\[ \leq \beta r \mu_n d(T^{n+1} x, y) + r(1 - \beta) \mu_n d(T^n x, y) \]

and hence

\[ \alpha \mu_n d(T^n x, Ty) + (1 - \alpha) \mu_n d(T^n x, Ty) \]

\[ \leq \beta r \mu_n d(T^n x, y) + r(1 - \beta) \mu_n d(T^n x, y). \]

Then we have that

\[ \mu_n d(T^n x, Ty) \leq r \mu_n d(T^n x, y) \]

for all \( y \in X \).
Defining a function $\psi : X \to [0, \infty)$ as in the proof of Theorem 3.2 by

$$\psi(y) = \frac{1}{1 - r} \mu_n d(T^n x, y), \quad \forall y \in X,$$

we have

$$\mu_n d(T^n x, y) + \psi(Ty) \leq \psi(y), \quad \forall y \in X.$$

Thus from Theorem 3.1, or Theorem 3.2, we obtain the desired result.
Theorem 3.4.

Assume that:

(i) \((X, d)\) is a complete metric space;

(ii) \(\psi : X \to (-\infty, \infty]\) is a proper, bounded below, and lower semicontinuous function;

(iii) Let \(T : X \to X\) be a mapping. Suppose that there exists \(\alpha \in \mathbb{R}\) such that
\[
\alpha d(Tx, y) + (1 - \alpha)d(x, y) + \psi(Ty) \leq \psi(y), \quad \forall x, y \in X.
\]

Then \(T\) has a unique fixed point \(\bar{x}\) in \(X\) such that
\[
\psi(\bar{x}) = \inf_{u \in X} \psi(u) \quad \text{and} \quad \bar{x} = \lim_{m \to \infty} T^m z \quad \text{for all} \quad z \in X \quad \text{with} \quad \psi(z) < \infty.
\]
Proof of Theorem 3.4.

Let us first consider $\alpha > 0$. By (iii), we have that

$$\alpha d(Tx, x) + \psi(Tx) \leq \psi(x), \quad \forall x \in X.$$  

From Theorem 2.1 (Caristi’s fixed point theorem), there exists $\bar{x} \in X$ such that $T\bar{x} = \bar{x}$. By (iii) again, we have that

$$d(\bar{x}, y) + \psi(Ty) \leq \psi(y), \quad \forall y \in X.$$  

By Theorem 3.1, we have that $\bar{x}$ is a unique fixed point of $T$ such that $\psi(\bar{x}) = \inf_{u \in X} \psi(u)$ and $\bar{x} = \lim_{m \to \infty} T^m z$ for all $z \in X$ with $\psi(z) < \infty$. 

Proof of Theorem 3.4-cont.

Next let us consider the case of $\alpha = 0$. Then we have

$$d(x, y) + \psi(Ty) \leq \psi(y), \quad \forall x, y \in X.$$ 

Replacing $x$ and $y$ by $Tx$ and $x$ in the above inequality respectively, we have that

$$d(Tx, x) + \psi(Tx) \leq \psi(x), \quad \forall x \in X.$$
Proof of Theorem 3.4-cont.

Next let us consider the case of $\alpha = 0$. Then we have
\[
d(x, y) + \psi(Ty) \leq \psi(y), \quad \forall x, y \in X.
\]
Replacing $x$ and $y$ by $Tx$ and $x$ in the above inequality respectively, we have that
\[
d(Tx, x) + \psi(Tx) \leq \psi(x), \quad \forall x \in X.
\]
By Theorem 2.1 (Caristi’s fixed point theorem), there exists $\bar{x} \in X$ such that $T\bar{x} = \bar{x}$. 
Proof of Theorem 3.4-cont.

Next let us consider the case of $\alpha = 0$. Then we have
\[
d(x, y) + \psi(Ty) \leq \psi(y), \quad \forall x, y \in X.
\]
Replacing $x$ and $y$ by $Tx$ and $x$ in the above inequality respectively, we have that
\[
d(Tx, x) + \psi(Tx) \leq \psi(x), \quad \forall x \in X.
\]
By Theorem 2.1 (Caristi’s fixed point theorem), there exists $\bar{x} \in X$ such that $T\bar{x} = \bar{x}$. Hence,
\[
d(\bar{x}, y) + \psi(Ty) \leq \psi(y), \quad \forall y \in X.
\]
Proof of Theorem 3.4-cont.

Next let us consider the case of $\alpha = 0$. Then we have

$$d(x, y) + \psi(Ty) \leq \psi(y), \quad \forall x, y \in X.$$  

Replacing $x$ and $y$ by $Tx$ and $x$ in the above inequality respectively, we have that

$$d(Tx, x) + \psi(Tx) \leq \psi(x), \quad \forall x \in X.$$  

By Theorem 2.1 (Caristi’s fixed point theorem), there exists $\bar{x} \in X$ such that $T\bar{x} = \bar{x}$. Hence,

$$d(\bar{x}, y) + \psi(Ty) \leq \psi(y), \quad \forall y \in X.$$  

By Theorem 3.1, we have that $\bar{x}$ is a unique fixed point of $T$ such that $\psi(\bar{x}) = \inf_{u \in X} \psi(u)$ and $\bar{x} = \lim_{m \to \infty} T^m z$ for all $z \in X$ with $\psi(z) < \infty$. 
Proof of Theorem 3.4-cont.

In the case of $\alpha < 0$, we have $1 - \alpha > 0$. By (iii),

$$(1 - \alpha)d(x, Tx) + \psi(T^2x) \leq \psi(Tx), \quad \forall x \in X.$$
Proof of Theorem 3.4-cont.

In the case of $\alpha < 0$, we have $1 - \alpha > 0$. By (iii),

$$(1 - \alpha)d(x, Tx) + \psi(T^2x) \leq \psi(Tx), \quad \forall x \in X.$$ 

Take $x \in X$ with $\psi(x) < \infty$. Then we have that for any $n \in \mathbb{N}$,

$$(1 - \alpha)d(x, Tx) + \psi(T^2x) \leq \psi(Tx),$$
$$(1 - \alpha)d(Tx, T^2x) + \psi(T^3x) \leq \psi(T^2x),$$
$$\vdots$$
$$(1 - \alpha)d(T^{n-1}x, T^nx) + \psi(T^{n+1}x) \leq \psi(T^nx).$$
Proof of Theorem 3.4-cont.

In the case of $\alpha < 0$, we have $1 - \alpha > 0$. By (iii),

$$(1 - \alpha)d(x, Tx) + \psi(T^2x) \leq \psi(Tx), \quad \forall x \in X.$$ 

Take $x \in X$ with $\psi(x) < \infty$. Then we have that for any $n \in \mathbb{N}$,

$$(1 - \alpha)d(x, Tx) + \psi(T^2x) \leq \psi(Tx),$$

$$(1 - \alpha)d(Tx, T^2x) + \psi(T^3x) \leq \psi(T^2x),$$

$$\vdots$$

$$(1 - \alpha)d(T^{n-1}x, T^nx) + \psi(T^{n+1}x) \leq \psi(T^nx).$$

Adding these inequalities, we have that

$$(1 - \alpha)\left\{d(x, Tx) + \cdots + d(T^{n-1}x, T^nx)\right\} \leq \psi(Tx) - \psi(T^{n+1}x).$$
Proof of Theorem 3.4-cont.

Since \( \{ \psi(T^n x) \} \) is a decreasing sequence, there exists
\[ s = \lim_{n \to \infty} \psi(T^n x). \]
Proof of Theorem 3.4-cont.

Since \( \{ \psi(T^n x) \} \) is a decreasing sequence, there exists

\[
s = \lim_{n \to \infty} \psi(T^n x).
\]

Thus we have that for any \( n \in \mathbb{N} \),

\[
(1 - \alpha)d(x, T^n x) \leq (1 - \alpha)\left\{ d(x, Tx) + \cdots + d(T^{n-1} x, T^n x) \right\} \\
\leq (1 - \alpha)\left\{ d(x, Tx) + \cdots + d(T^{n-1} x, T^n x) + \cdots \right\} \\
\leq \psi(Tx) - s < \infty.
\]

Then \( \{ T^n x \} \) is bounded.
Proof of Theorem 3.4-cont.

Since \( \{\psi(T^n x)\} \) is a decreasing sequence, there exists \( s = \lim_{n \to \infty} \psi(T^n x) \). Thus we have that for any \( n \in \mathbb{N} \),

\[
(1 - \alpha) d(x, T^n x) \leq (1 - \alpha) \{ d(x, Tx) + \cdots + d(T^{n-1} x, T^n x) \} \\
\leq (1 - \alpha) \{ d(x, Tx) + \cdots + d(T^{n-1} x, T^n x) + \cdots \} \\
\leq \psi(Tx) - s < \infty.
\]

Then \( \{T^n x\} \) is bounded. By (iii) again, we have that for any \( n \in \mathbb{N} \),

\[
\alpha d(T^{n+1} x, y) + (1 - \alpha) d(T^n x, y) + \psi(Ty) \leq \psi(y), \quad \forall y \in X.
\]
Proof of Theorem 3.4-cont.

Since \( \{ \psi(T^n x) \} \) is a decreasing sequence, there exists
\[
s = \lim_{n \to \infty} \psi(T^n x).
\]
Thus we have that for any \( n \in \mathbb{N} \),
\[
(1 - \alpha)d(x, T^n x) \leq (1 - \alpha)\{ d(x, Tx) + \cdots + d(T^{n-1}x, T^n x) \}
\leq (1 - \alpha)\{ d(x, Tx) + \cdots + d(T^{n-1}x, T^n x) + \cdots \}
\leq \psi(Tx) - s < \infty.
\]
Then \( \{ T^n x \} \) is bounded. By (iii) again, we have that for any \( n \in \mathbb{N} \),
\[
\alpha d(T^{n+1} x, y) + (1 - \alpha)d(T^n x, y) + \psi(Ty) \leq \psi(y), \quad \forall y \in X.
\]
Applying a Banach limit \( \mu \) to the both sides of this inequality, we have that
\[
\alpha \mu_n d(T^{n+1} x, y) + (1 - \alpha)\mu_n d(T^n x, y) + \psi(Ty) \leq \psi(y), \quad \forall y \in X.
\]
Then we get that

\[ \mu_n d(T^n x, y) + \psi(Ty) \leq \psi(y), \quad \forall y \in X. \]

From Theorem 3.1, we have that \( T \) has a unique fixed point \( \bar{x} \) in \( X \) such that \( \psi(\bar{x}) = \inf_{u \in X} \psi(u) \) and \( \bar{x} = \lim_{m \to \infty} T^m z \) for all \( z \in X \) with \( \psi(z) < \infty \).
Theorem 3.5.

Assume that:

(i) \((X, d)\) is a complete metric space;

(ii) \(\psi : X \rightarrow (-\infty, \infty]\) is a proper, bounded below, and lower semicontinuous function;

(iii) Let \(T : X \rightarrow X\) be a mapping. Suppose that there exists \(\alpha \in \mathbb{R}\) such that

\[
\alpha d(Tx, Ty) + (1 - \alpha)d(x, Ty) + \psi(Ty) \leq \psi(y), \quad \forall x, y \in X.
\]

Then \(T\) has a unique fixed point \(\bar{x}\) in \(X\) such that

\[
\psi(\bar{x}) = \inf_{u \in X} \psi(u) \quad \text{and} \quad \bar{x} = \lim_{m \rightarrow \infty} T^m z \quad \text{for all} \quad z \in X \quad \text{with} \quad \psi(z) < \infty.
\]
Proof of Theorem 3.5-cont.

Let us first consider $\alpha > 0$. By (iii), we have that

$$\alpha d(T^2x, Tx) + \psi(Tx) \leq \psi(x), \quad \forall x \in X.$$ 

As in the proof of Theorem 3.4, we have that $T$ has a unique fixed point $\bar{x}$ in $X$ such that $\psi(\bar{x}) = \inf_{u \in X} \psi(u)$ and $\bar{x} = \lim_{m \to \infty} T^m z$ for all $z \in X$ with $\psi(z) < \infty$. 

In the case of $\alpha = 0$. By (iii) again, we have that

$$d(x, Tx) + \psi(Tx) \leq \psi(x), \quad \forall x \in X.$$ 

Similarly, in the case of $\alpha < 0$. By (iii) again, we have that

$$(1 - \alpha)d(x, Tx) + \psi(Tx) \leq \psi(x), \quad \forall x \in X.$$ 

As in the proof of Theorem 3.4 for such two cases, we get the conclusion.
Proof of Theorem 3.5-cont.

Let us first consider $\alpha > 0$. By (iii), we have that 
\[
\alpha d(T^2x, Tx) + \psi(Tx) \leq \psi(x), \quad \forall x \in X.
\]

As in the proof of Theorem 3.4, we have that $T$ has a unique fixed point $\bar{x}$ in $X$ such that $\psi(\bar{x}) = \inf_{u \in X} \psi(u)$ and $\bar{x} = \lim_{m \to \infty} T^m z$ for all $z \in X$ with $\psi(z) < \infty$.

In the case of $\alpha = 0$. By (iii) again, we have that 
\[
d(x, Tx) + \psi(Tx) \leq \psi(x), \quad \forall x \in X.
\]
Proof of Theorem 3.5-cont.

Let us first consider $\alpha > 0$. By (iii), we have that
\[ \alpha d(T^2x, Tx) + \psi(Tx) \leq \psi(x), \quad \forall x \in X. \]

As in the proof of Theorem 3.4, we have that $T$ has a unique fixed point $\bar{x}$ in $X$ such that $\psi(\bar{x}) = \inf_{u \in X} \psi(u)$ and $\bar{x} = \lim_{m \to \infty} T^m z$ for all $z \in X$ with $\psi(z) < \infty$.

In the case of $\alpha = 0$. By (iii) again, we have that
\[ d(x, Tx) + \psi(Tx) \leq \psi(x), \quad \forall x \in X. \]

Similarly, in the case of $\alpha < 0$. By (iii) again, we have that
\[ (1 - \alpha)d(x, Tx) + \psi(Tx) \leq \psi(x), \quad \forall x \in X. \]

As in the proof of Theorem 3.4 for such two cases, we get the conclusion.
4. Fixed Point Theorems for Set-Valued Mappings
Theorem 4.1.

Assume that:

(a) \((X, d)\) is a complete metric space;
(b) \(\{x_n\}\) is a bounded sequence in \(X\);
(c) \(\mu\) is a mean on \(\ell^\infty\) and let \(\psi : X \to (-\infty, \infty]\) be a proper, bounded below, and lower semicontinuous function;
(d) \(T : X \to P(X)\) is a set-valued mapping such that for each \(u \in X\), there exists \(v \in Tu\) satisfying

\[ \mu_n d(x_n, u) + \psi(v) \leq \psi(u). \]

Then there exists \(\bar{x} \in X\) such that
Theorem 4.1.

Assume that:

(a) \((X, d)\) is a complete metric space;
(b) \(\{x_n\}\) is a bounded sequence in \(X\);
(c) \(\mu\) is a mean on \(\ell^\infty\) and let \(\psi : X \to (-\infty, \infty]\) be a proper, bounded below, and lower semicontinuous function;
(d) \(T : X \to P(X)\) is a set-valued mapping such that for each \(u \in X\), there exists \(v \in Tu\) satisfying

\[
\mu_n d(x_n, u) + \psi(v) \leq \psi(u).
\]

Then there exists \(\bar{x} \in X\) such that

(i) \(\mu_n d(x_n, \bar{x}) = 0\);
Theorem 4.1.

Assume that:

(a) \((X, d)\) is a complete metric space;
(b) \(\{x_n\}\) is a bounded sequence in \(X\);
(c) \(\mu\) is a mean on \(\ell^\infty\) and let \(\psi : X \to (-\infty, \infty]\) be a proper, bounded below, and lower semicontinuous function;
(d) \(T : X \to P(X)\) is a set-valued mapping such that for each \(u \in X\), there exists \(v \in Tu\) satisfying

\[\mu_n d(x_n, u) + \psi(v) \leq \psi(u).\]

Then there exists \(\bar{x} \in X\) such that

(i) \(\mu_n d(x_n, \bar{x}) = 0\);  (ii) \(\bar{x} \in T\bar{x}\);
Theorem 4.1.

Assume that:
(a) \((X, d)\) is a complete metric space;
(b) \(\{x_n\}\) is a bounded sequence in \(X\);
(c) \(\mu\) is a mean on \(\ell^\infty\) and let \(\psi : X \to (-\infty, \infty]\) be a proper, bounded below, and lower semicontinuous function;
(d) \(T : X \to P(X)\) is a set-valued mapping such that for each \(u \in X\), there exists \(v \in Tu\) satisfying
\[
\mu_n d(x_n, u) + \psi(v) \leq \psi(u).
\]

Then there exists \(\bar{x} \in X\) such that
(i) \(\mu_n d(x_n, \bar{x}) = 0\);
(ii) \(\bar{x} \in T\bar{x}\);
(iii) \(\psi(\bar{x}) = \inf_{y \in X} \psi(y)\);
Theorem 4.1.

Assume that:
(a) $(X, d)$ is a complete metric space;
(b) $\{x_n\}$ is a bounded sequence in $X$;
(c) $\mu$ is a mean on $\ell^\infty$ and let $\psi : X \to (-\infty, \infty]$ be a proper, bounded below, and lower semicontinuous function;
(d) $T : X \to P(X)$ is a set-valued mapping such that for each $u \in X$, there exists $v \in Tu$ satisfying
$$\mu_n d(x_n, u) + \psi(v) \leq \psi(u).$$

Then there exists $\bar{x} \in X$ such that
(i) $\mu_n d(x_n, \bar{x}) = 0$;  
(ii) $\bar{x} \in T\bar{x}$;  
(iii) $\psi(\bar{x}) = \inf_{y \in X} \psi(y)$;
(iv) for any $y \in X$ with $\psi(y) < \infty$, there exists a sequence $\{y_n\} \subset X$ such that $y_{n+1} \in Ty_n$, $n \in \mathbb{N} \cup \{0\}$ and $y_n \to \bar{x}$.
Proof of Theorem 4.1.

For each \( u_1 \in X \) with \( \psi(u_1) < \infty \), there exists \( u_2 \in Tu_1 \) such that
\[
\mu_n d(x_n, u_1) \leq \psi(u_1) - \psi(u_2).
\]
Repeating this process, we get a sequence \( \{u_m\} \) in \( X \) such that
\( u_{m+1} \in Tu_m \) and
\[
\mu_n d(x_n, u_m) \leq \psi(u_m) - \psi(u_{m+1})
\]
for each \( m \in \mathbb{N} \). Clearly, \( \{\psi(u_m)\} \) is a decreasing sequence.

Furthermore, we have:
\[
\lim_{m \to \infty} \mu_n d(x_n, u_m) = 0.
\]
Hence for each \( \varepsilon > 0 \), there exists \( m_0 \in \mathbb{N} \) such that for any \( m \geq m_0 \),
\[
\mu_n d(x_n, u_m) < \frac{\varepsilon}{2}.
\]
Proof of Theorem 4.1-cont.

This implies that for all \( m_1, m_2 \geq m_0 \),

\[
d(u_{m_1}, u_{m_2}) \leq \mu_n d(x_n, u_{m_1}) + \mu_n d(x_n, u_{m_2}) < \varepsilon.
\]

Then \( \{u_m\} \) is a Cauchy sequence in \( X \), and there exists \( \bar{x} \in X \) such that \( \lim_{m \to \infty} u_m = \bar{x} \) and \( \mu_n d(x_n, \bar{x}) = 0 \). By assumption again, we have \( \bar{u} \in X \) such that \( \bar{u} \in T\bar{x} \) and

\[
\mu_n d(x_n, \bar{x}) \leq \psi(\bar{x}) - \psi(\bar{u}).
\]

Then we have \( \psi(\bar{u}) \leq \psi(\bar{x}) \). Furthermore, repeating this process, we have \( \bar{v} \in X \) such that \( \bar{v} \in T\bar{u} \) and

\[
\mu_n d(x_n, \bar{u}) \leq \psi(\bar{u}) - \psi(\bar{v}).
\]

Then we have \( \psi(\bar{v}) \leq \psi(\bar{u}) \).
Proof of Theorem 4.1-cont.

Next, following the same argument as above for each $y_1 \in X$ with $\psi(y_1) < \infty$, we can construct a sequence $\{y_m\}$ in $X$ such that

$$\lim_{m \to \infty} \psi(y_m) = \inf_{m \in \mathbb{N}} \psi(y_m) \text{ exists, } \lim_{m \to \infty} y_m = \bar{y} \text{ for some } \bar{y} \in X,$$

and $\mu_n d(x_n, \bar{y}) = 0$. Since $d(\bar{x}, \bar{y}) \leq d(\bar{y}, x_n) + d(x_n, \bar{x})$, we have

$$d(\bar{x}, \bar{y}) \leq \mu_n d(\bar{y}, x_n) + \mu_n d(x_n, \bar{x}) = 0 + 0 = 0.$$

Thus we have $\bar{y} = \bar{x}$. Since $\psi$ is lower semicontinuous,

$$\psi(\bar{x}) = \psi(\bar{y}) \leq \liminf_{m \to \infty} \psi(y_m) = \lim_{m \to \infty} \psi(y_m) = \inf_{m \in \mathbb{N}} \psi(y_m) \leq \psi(y_1).$$

Since $y_1$ is any point of $X$ with $\psi(y_1) < \infty$, we have that

$$\psi(\bar{x}) = \inf_{x \in X} \psi(x) \leq \psi(\bar{v}) \leq \psi(\bar{u}) \leq \psi(\bar{x}).$$

Thus we have $\psi(\bar{x}) = \psi(\bar{v}) = \psi(\bar{u})$. This implies that $\bar{u} = \bar{x}$.

Therefore we have from $\bar{u} \in T\bar{x}$ that $\bar{x} = \bar{u} \in T\bar{x}$. 
Notations.

Let \((X, d)\) be a metric space and let \(P(X)\) be the set of nonempty subsets of \(X\).

Then \(T : X \to P(X)\) is called a multi-valued weakly Picard operator [17] if for each \(x \in X\) and each \(y \in Tx\), there exists a sequence \(\{x_n\}\) in \(X\) such that

1. \(x_0 = x, \ x_1 = y\);
2. \(x_{n+1} \in Tx_n, \ n \in \mathbb{N} \cup \{0\}\);
3. \(\{x_n\}\) is convergent and its limit is a fixed point of \(T\).
Theorem 4.2.

Assume that:

(a) \((X, d)\) is a complete metric space;
(b) \(\{x_n\}\) is a bounded sequence in \(X\);
(c) \(\mu\) is a mean on \(\ell^\infty\);
(d) \(\psi : X \to (-\infty, \infty)\) is a bounded below and lower semicontinuous function;
(e) Let \(T : X \to P(X)\) be a set-valued mapping such that for each \(u \in X\), there exists \(v \in Tu\) satisfying 

\[
\mu_n d(x_n, u) + \psi(v) \leq \psi(u).
\]

Then \(T\) is a multi-valued weakly Picard operator.
Proof of Theorem 4.2.

For each \( x \in X \) and each \( y \in Tx \), put \( u_0 = x \) and \( u_1 = y \). Since \( \psi \) is a real-valued function on \( X \), we can take \( u_2 \in Tu_1 \) such that

\[
\mu_n d(x_n, u_1) + \psi(u_2) \leq \psi(u_1).
\]

Repeating this process, we get a sequence \( \{u_m\} \) in \( X \) such that \( u_{m+1} \in Tu_m \) and

\[
\mu_n d(x_n, u_m) \leq \psi(u_m) - \psi(u_{m+1})
\]

for each \( m \in \mathbb{N} \cup \{0\} \). Thus we have the desired result from Theorem 4.1.
Theorem 4.3.

Assume that:

(a) \((X, d)\) is a complete metric space;

(b) \(T : X \to BC(X)\) is an \(\alpha\)-contraction with \(0 \leq \alpha < 1\) such that \(SF(T) \neq \emptyset\).

Then the following hold:

(i) \(F(T) = SF(T) = \{x^*\}\);

(ii) for any \(x \in X\), there exists \(\{x_n\} \subset X\) such that \(x_0 = x\), \(x_{n+1} \in Tx_n\) for all \(n \in \mathbb{N} \cup \{0\}\) and \(x_n \to x^*\).
Proof of Theorem 4.3.

(i) See [16] and [15]. In fact, let $x^* \in SF(T)$. Then we have

$$\{x^*\} \subset SF(T) \subset F(T).$$

Conversely, if $u \in F(T)$, then we have that

$$d(u, x^*) \leq \sup_{v \in Tu} d(v, x^*) = \delta(Tu, x^*)$$

$$= \delta(Tu, Tx^*) \leq H(Tu, Tx^*) \leq \alpha d(u, x^*).$$

Since $0 \leq \alpha < 1$, we have $d(u, x^*) = 0$ and hence $u = x^*$. Thus we have that

$$\{x^*\} = SF(T) = F(T).$$
Proof of Theorem 4.3-cont.

(ii) We have that for any $x \in X$ and $u \in Tx$,

$$d(x, x^*) \leq d(x, u) + d(u, x^*) \leq d(x, u) + H(Tx, x^*).$$
Proof of Theorem 4.3-cont.

(ii) We have that for any $x \in X$ and $u \in Tx$,

\[ d(x, x^*) \leq d(x, u) + d(u, x^*) \leq d(x, u) + H(Tx, x^*). \]

Thus we have that

\[ d(x, x^*) \leq d(x, Tx) + H(Tx, x^*) \leq d(x, Tx) + \alpha d(x, x^*) \]
Proof of Theorem 4.3-cont.

(ii) We have that for any \( x \in X \) and \( u \in Tx \),

\[
d(x, x^*) \leq d(x, u) + d(u, x^*) \leq d(x, u) + H(Tx, x^*).
\]

Thus we have that

\[
d(x, x^*) \leq d(x, Tx) + H(Tx, x^*) \leq d(x, Tx) + \alpha d(x, x^*)
\]

\( \Rightarrow \) \( (1 - \alpha)d(x, x^*) \leq d(x, Tx) \).
(ii) We have that for any \( x \in X \) and \( u \in Tx \),

\[
d(x, x^*) \leq d(x, u) + d(u, x^*) \leq d(x, u) + H(Tx, x^*).
\]

Thus we have that

\[
d(x, x^*) \leq d(x, Tx) + H(Tx, x^*) \leq d(x, Tx) + \alpha d(x, x^*)
\]

\[
\Rightarrow (1 - \alpha) d(x, x^*) \leq d(x, Tx).
\]

Choose a positive \( \varepsilon \) with \( \varepsilon < \frac{1}{\alpha} - 1 \). For any \( u \in X \) with \( u \neq x^* \), we have that \( d(u, x^*) > 0 \) and \( d(u, Tu) > 0 \).
Proof of Theorem 4.3-cont.

(ii) We have that for any $x \in X$ and $u \in Tx$,

$$d(x, x^*) \leq d(x, u) + d(u, x^*) \leq d(x, u) + H(Tx, x^*).$$

Thus we have that

$$d(x, x^*) \leq d(x, Tx) + H(Tx, x^*) \leq d(x, Tx) + \alpha d(x, x^*)$$

$$\Rightarrow (1 - \alpha)d(x, x^*) \leq d(x, Tx).$$

Choose a positive $\varepsilon$ with $\varepsilon < \frac{1}{\alpha} - 1$. For any $u \in X$ with $u \neq x^*$, we have that $d(u, x^*) > 0$ and $d(u, Tu) > 0$. Furthermore, we can take $v \in Tu$ from the above inequality such that

$$d(u, v) \leq d(u, Tu) + \varepsilon \left\{ d(u, Tu) - \frac{1}{2}(1 - \alpha)d(x^*, u) \right\}$$

$$= (1 + \varepsilon)d(u, Tu) - \frac{1}{2}\varepsilon(1 - \alpha)d(x^*, u).$$
Proof of Theorem 4.3-cont.

On the other hand, since $T$ is an $\alpha$-contraction, we have that

$$d(v, Tv) \leq H(Tu, Tv)$$

$$\leq \alpha d(u, v)$$

$$\leq \alpha \{(1 + \varepsilon)d(u, Tu) - \frac{1}{2}\varepsilon(1 - \alpha)d(x^*, u)\} \quad (2)$$

$$= \alpha(1 + \varepsilon)d(u, Tu) - \frac{1}{2}\alpha\varepsilon(1 - \alpha)d(x^*, u)$$

$$\leq \alpha(1 + \varepsilon)d(u, Tu).$$

Thus we have:

$$d(u, Tu) - d(v, Tv) \geq d(u, Tu) - \alpha d(u, v)$$

$$\geq \frac{1}{1+\varepsilon}\{d(u, v) + \frac{1}{2}\varepsilon(1 - \alpha)d(x^*, u)\} - \alpha d(u, v)$$

$$= \left(\frac{1}{1+\varepsilon} - \alpha\right)d(u, v) + \frac{\varepsilon}{2(1+\varepsilon)}(1 - \alpha)d(x^*, u).$$
Proof of Theorem 4.3-cont.

So, we can choose a sequence \( \{x_n\} \subset X \) such that \( x_0 = u \),
\( x_{n+1} \in Tx_n \),

\[
d(x_n, Tx_n) - d(x_{n+1}, Tx_{n+1}) \geq \left( \frac{1}{1 + \varepsilon} - \alpha \right) d(x_n, x_{n+1})
\]
and

\[
d(x_n, Tx_n) - d(x_{n+1}, Tx_{n+1}) \geq \frac{\varepsilon}{2(1 + \varepsilon)} (1 - \alpha) d(x^*, x_n).
\]

Then \( \{x_n\} \) is a Caucy sequence, and \( \{x_n\} \) converges to \( x^* \). This completes the proof.
References


Thank you very much for your attention!