Risk neutral valuation of options in multi-period discrete-time markets

In this lecture we study multi-period markets and show the fundamental theorem that such markets are arbitrage-free if and only there exists a martingale measure, that is, a probability with respect to which the discounted asset price processes are martingales.

We also show that the price of an attainable contingent claim is the discounted expectation of its payoff under a martingale measure and use it to price call and put options in the binomial model.

Partitions, Algebras and Filtrations

Let $\Omega = \{\omega_1, \omega_2, \ldots\}$ be the *sample space*, which we assume to be a finite set.

Definition. A *partition* of Ω is a collection of subsets (or *events*) Ω_1, Ω_2,…,Ω_m such that $\Omega_i \cap \Omega_j$ is empty if $i \neq j$ and $\Omega = \bigcup_{i=1}^{m} \Omega_i$.

Definition (*Algebra generated by a partition*). If $\mathcal{P} = \{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ is a partition of Ω, we define $\mathcal{F}$ to be the collection of finite unions of the Ω_i. Then $\mathcal{F}$ is an *algebra* of sets, that is,

$$\Omega \in \mathcal{F}, \quad A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F},$$

$$A \in \mathcal{F},\ B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}.$$

Definition. An increasing sequence of algebras $\{\mathcal{F}_n\}_{0 \le n \le N}$ of subsets of Ω, that is,

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_N,$$

is called a *filtration*. Note we always assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Usually, $\mathcal{F}_N$ is all subsets of Ω but not always.

Random Variables

Let $\mathcal{F}$ be an algebra of subsets of the finite sample space Ω.

Definition. A real-valued function (or *random variable*) $X : \Omega \to \mathcal{R}$ is said to be $\mathcal{F}$-*measurable* if for all $x \in \mathcal{R}$ the set $\{\omega \in \Omega : X(\omega) = x\}$ is in $\mathcal{F}$.

Definition. Let X be a random variable on Ω. Then X generates a partition $\mathcal{P}_X = \{\Omega_1, ..., \Omega_n\}$ of Ω as follows: let x_1,....,x_n be the distinct values taken by X. For $i = 1, ..., n$, define

$$\Omega_i = \{\omega \in \Omega | X(\omega) = x_i\}.$$

$\mathcal{P}_X$ can be thought of as the information carried by X. We denote by $\mathcal{F}_X$ the algebra generated by $\mathcal{P}_X$.

If $\mathcal{F}$ is an algebra in Ω, a random variable X is measurable with respect to $\mathcal{F}$ if and only if $\mathcal{F}_X \subset \mathcal{F}$.

Stochastic Processes

Definition. A real-valued *stochastic process* is a sequence of random variables $\{X_n\}_{0 \le n \le N}$ on Ω. We say it is *adapted* to a filtration $\{\mathcal{F}_n\}_{0 \le n \le N}$ if X_n is $\mathcal{F}_n$-measurable for each n.

Definition. If $\{X_n\}_{0 \le n \le N}$ is a stochastic process, then we define $\mathcal{P}_n$ to be the partition given by the sets $\bigcap_{i=1}^{n} A_i$, where $A_i \in \mathcal{P}_{X_i}$. Then if $\mathcal{F}_n$ is the corresponding algebra, we say that $\{\mathcal{F}_n\}_{0 \le n \le N}$ is the *natural filtration* associated with the stochastic process $\{X_n\}_{0 \le n \le N}$.

Fact. $\mathcal{F}_n$ as just defined is the smallest algebra with respect to which X_0, X_1,..., X_n are measurable.

Conditional Expectation

Let Ω be a finite sample space.

Definition. A *probability measure* on Ω is a function $P : \Omega \to \mathcal{R}$ satisfying $P(\omega_i) \geq 0$ for all i and

$$\sum P(\omega_i) = 1.$$

We write $P(\omega_i) = p_i$. If $A \subset \Omega$, we define

$$P(A) = \sum_{\omega_i \in A} p_i.$$

Definition. If $X : \Omega \to \mathcal{R}$ is a random variable, we define its *expected value* to be

$$E(X) = \sum p_i X(\omega_i) = \sum x P(X = x).$$

If $X : \Omega \to \mathcal{R}$ and $Y : \Omega \to \mathcal{R}$ are random variables, we define the conditional expectation of X with respect to Y by

$$E(X|Y = y) = \sum xP(X = x|Y = y).$$

If $A = \{\omega : Y(\omega) = y\}$, we can think of this as $E(X|A)$. Now we write this as $E(X|\mathcal{F}_Y)(\omega)$ for ω in A.

Definition. Let $\mathcal{F}$ be an algebra of subsets of Ω with associated partition $\mathcal{P}$. Then we define the *conditional expectation* of a random variable X with respect to $\mathcal{F}$ to be the random variable defined by

$$E(X|\mathcal{F})(\omega) = \frac{\sum_{\omega_i \in A} p_i X(\omega_i)}{P(A)} = \sum xP(X = x|A)$$

if $\omega \in A$ and $A \in \mathcal{P}$. Note that $E(X|\mathcal{F})$ is measurable with respect to $\mathcal{F}$.

Properties of conditional expectation

1. (*Tower Property*) If $\mathcal{F}$ and $\mathcal{G}$ are algebras with $\mathcal{F} \subset \mathcal{G}$, then

$$E(E(X|\mathcal{G})|\mathcal{F}) = E(X|\mathcal{F}).$$

2. If X and Y are random variables and Y is $\mathcal{F}$-measurable, then

$$E(XY|\mathcal{F}) = YE(X|\mathcal{F}).$$

In particular, $E(Y|\mathcal{F}) = Y$.

3. If the random variable X is *independent* of $\mathcal{F}$ (that is,

$$P(X(\omega) = x \text{ and } \omega \in A) = P(X(\omega) = x) \times P(A)$$

for all $A \in \mathcal{F}$), then

$$E(X|\mathcal{F}) = E(X).$$

Now we define the conditional expectation of a random variable, given another random variable.

Definition. Let X and Y be random variables. Then we define the random variable

$$E(X|Y) = E(X|\mathcal{F}_Y).$$

So if $A = \{\omega : Y(\omega) = y\}$ and $\omega \in A$,

$$E(X|Y)(\omega) = \sum xP(X = x|A) = \sum xP(X = x|Y = y)$$

the usual thing.

Martingales

Definition. Suppose that $(\Omega, \{\mathcal{F}_n\}_{n=0}^N, P)$ is a *filtered probability space*, that is, (Ω, P) is a (finite) probability space and $\{\mathcal{F}_n\}_{n=0}^N$ is a filtration.

The stochastic process $\{X_n\}_{n=0}^N$ is said to be a *martingale* with respect to P if it is adapted to the filtration $\{\mathcal{F}_n\}_{n=0}^N$ and for all n

$$E(X_{n+1}|\mathcal{F}_n) = X_n.$$

If for all n

$$E(X_{n+1}|\mathcal{F}_n) \leq X_n,$$

$\{X_n\}_{n=0}^N$ is said to be a *supermartingale*. If for all n

$$E(X_{n+1}|\mathcal{F}_n) \geq X_n,$$

$\{X_n\}_{n=0}^N$ is said to be a *submartingale*.

Fundamental Theorem of Asset Pricing

Now we assume trading takes place at times

$$0 = t_0 < t_1 < \cdots < t_N = T,$$

where $t_n = n\delta t$.

Let the stock price process be $\{S_n\}_{0 \leq n \leq N}$, which is a quite general stochastic process on a finite probability space (Ω, P), where we assume $P(\omega) > 0$ for all $\omega \in \Omega$. We can think of Ω as the possible paths followed by S_n. S_n is the stock price at time $n\delta t$.

Let $\{\mathcal{F}_n\}_{0 \leq n \leq N}$ be the natural filtration associated with $\{S_n\}_{0 \leq n \leq N}$. Each event A in the partition $\mathcal{P}_n$ associated with $\mathcal{F}_n$ corresponds to a stock price path up to time n. Note that usually (but not necessarily) $\mathcal{F}_N$ consists of all subsets of Ω.

The market has just two securities, the other being the risk-free bond, where it is assumed that a bond at $t = 0$ is worth $B_n > 0$ at time $n\delta t$ with $B_0 = 1$. (B_n is not stochastic.)

Let $\{(\phi_n, \psi_n)\}_{1 \leq n \leq N}$ be a portfolio of ϕ_n shares and ψ_n bonds. So (ϕ_n, ψ_n) is the amount of stock and bond held in the portfolio over the time interval $[(n-1)\delta t, n\delta t)$.

The *value* of the portfolio at time $n\delta t$ is

$$V_n = \phi_{n+1} S_n + \psi_{n+1} B_n$$

for $n = 0, \ldots, N-1$ and

$$V_N = \phi_N S_N + \psi_N B_N.$$

The portfolio is *self-financing* if for $n = 1, \ldots, N-1$

$$\phi_n S_n + \psi_n B_n = \phi_{n+1} S_n + \psi_{n+1} B_n.$$

Definition. An *arbitrage opportunity* in a multi-period securities market is a self-financing portfolio $\{(\phi_n, \psi_n)\}_{1 \le n \le N}$ such that its value process $\{V_n\}_{0 \le n \le N}$ satisfies either

$$V_0 < 0, \quad V_N \ge 0$$

or

$$V_0 = 0, \quad V_N \ge 0, \quad E(V_N) > 0.$$

Proposition. *Let Q be a measure on Ω such that the discounted price process $\{\tilde{S}_n = B_n^{-1} S_n\}_{0 \le n \le N}$ is a $(Q, \{\mathcal{F}_n\}_{0 \le n \le N})$-martingale, that is,*

$$E^Q(\tilde{S}_{n+1} | \mathcal{F}_n) = \tilde{S}_n, \quad n = 0, \ldots, N-1.$$

Then the discounted value process $\{\tilde{V}_n\}_{0 \le n \le N}$ of a self-financing portfolio $\{(\phi_n, \psi_n)\}_{1 \le n \le N}$ is also a martingale.

Proof. The discounted value process is

$$\tilde{V}_n = B_n^{-1} V_n = \phi_{n+1}\tilde{S}_n + \psi_{n+1}.$$

Using the self-financing property,

$$\begin{aligned}\tilde{V}_{n+1} - \tilde{V}_n &= \phi_{n+2}\tilde{S}_{n+1} + \psi_{n+2} - \phi_{n+1}\tilde{S}_n - \psi_{n+1} \\ &= \phi_{n+1}\tilde{S}_{n+1} + \psi_{n+1} - \phi_{n+1}\tilde{S}_n - \psi_{n+1} \\ &= \phi_{n+1}(\tilde{S}_{n+1} - \tilde{S}_n).\end{aligned}$$

Next since S_n is adapted and ϕ_n is previsible, $\tilde{V}_n$ is also adapted. Then for $n \geq 0$

$$\begin{aligned}E(\tilde{V}_{n+1}|\mathcal{F}_n) - \tilde{V}_n &= E(\tilde{V}_{n+1} - \tilde{V}_n|\mathcal{F}_n) \\ &= E(\phi_{n+1}(\tilde{S}_{n+1} - \tilde{S}_n)|\mathcal{F}_n) \\ &= \phi_{n+1}E(\tilde{S}_{n+1} - \tilde{S}_n|\mathcal{F}_n) \\ &= \phi_{n+1}[E(\tilde{S}_{n+1}|\mathcal{F}_n) - \tilde{S}_n] \\ &= 0.\end{aligned}$$

Definition. A measure Q on Ω such that $Q(\omega) > 0$ for all ω and such that the discounted stock price process $\{\tilde{S}_n\}_{0\leq n\leq N}$ is a $(Q, \{\mathcal{F}_n\}_{0\leq n\leq N})$-martingale is called a *martingale measure*.

Fundamental Theorem of Asset Pricing. *For the multiperiod market described above, there is no arbitrage if and only if there is a martingale measure.*

Proof. Suppose first there exists a martingale measure Q. Consider a self-financing portfolio $\{(\phi_n, \psi_n)\}_{1 \le n \le N}$ with value process satisfying

$$V_0 \le 0, \quad V_N \ge 0.$$

Then, by the Proposition, we know that the discounted value process $\{\tilde{V}_n\}_{0 \le n \le N}$ is also a $(Q, \{\mathcal{F}_n\}_{0 \le n \le N})$-martingale. Then

$$V_0 = E^Q(\tilde{V}_1|\mathcal{F}_0) = E^Q(E^Q(\tilde{V}_2|\mathcal{F}_1)|\mathcal{F}_0) = E^Q(\tilde{V}_2|\mathcal{F}_0).$$

Repeating this argument (which holds for any martingale),

$$V_0 = E^Q(\tilde{V}_N|\mathcal{F}_0).$$

Since $\mathcal{F}_0 = \{\emptyset, \Omega\}$, this means that

$$V_0 = E^Q(\tilde{V}_N).$$

Since $V_0 \le 0$, $V_N \ge 0$, it follows that $V_0 = E^Q(\tilde{V}_N) = 0$, which implies $E^Q(V_N) = 0$. So there can be no arbitrage opportunity.

Now suppose, conversely, that our market has no arbitrage opportunity. We want to show the existence of a martingale measure. We use the result for one-period models.

To this end fix a time n and let A be any set in the partition $\mathcal{P}_n$ corresponding to $\mathcal{F}_n$. Since S_k is $\mathcal{F}_k$-measurable for $0 \le k \le n$, it follows that S_k is constant on A for $0 \le k \le n$. Hence if the event A is known to have occurred, $S_k(\omega)$ are known for $0 \le k \le n$, that is, the path of the stock price process up to time n is known. However $S_{n+1}(\omega)$ may not be constant over A. Let $\{s_1, s_2, \ldots, s_m\}$ be its possible values and put

$$A_j = \{\omega \in A : S_{n+1}(\omega) = s_j\}.$$

Note that $A_j \in \mathcal{P}_{n+1}$, the partition corresponding to $\mathcal{F}_{n+1}$. If $\omega \in A$, we also put

$$s_0 = S_n(\omega).$$

So now we have a one-period securities market with two securities, the stock and the risk-free security (if a bond is bought for \$1, it pays $B_{n+1}B_n^{-1}$ at the end of the period so that the discount factor is $B_{n+1}^{-1}B_n$). There are m states of the world.

We show now that *this one-period market has no arbitrage opportunity.* For suppose it did. Then it would yield an arbitrage opportunity in the multi-period market as follows: Let ϕ be the number and ψ the number of bonds in the portfolio giving the arbitrage opportunity. Then we define

$$\phi_k(\omega) = \psi_k(\omega) = 0$$

for $1 \leq k \leq N$ if ω is not in A and for $1 \leq k \leq n$ if ω is in A. Next if $\omega \in A$, we define

$$\phi_{n+1} = \phi, \quad \psi_{n+1} = -B_n^{-1}\phi s_0$$

and for $n + 2 \leq k \leq N$

$$\phi_k = 0, \quad \psi_k = \phi(S_{n+1}B_{n+1}^{-1} - s_0 B_n^{-1}).$$

It is easy to show this is an arbitrage opportunity in the multi-period market with initial zero cost $V_0 = 0$, $V_N \geq 0$ and positive probability that $V_N > 0$.

So we have shown that *if the multi-period market has no arbitrage opportunity, then for all* n *and* $A \in \mathcal{P}_n$ *the one period market with the current stock price* $s_0 = S_n(\omega)$ *and future stock prices* $\{s_1, s_2, \dots, s_m\}$*, where* $s_j = S_{n+1}(\omega)$ *for* $\omega \in A_j \in \mathcal{P}_{n+1}$*, has no arbitrage opportunity also.*

Then we know there exist risk-neutral probabilities $p_j > 0$, $j = 1, \dots, m$, such that $p_1 + p_2 + \cdots + p_m = 1$ and

$$s_0 = B_{n+1}^{-1} B_n (p_1 s_1 + p_2 s_2 + \cdots + p_m s_m).$$

Then for $j = 1, \dots, m$ we define

$$q(A_j, A) = p_j.$$

Note that $q(\cdot,\cdot)$ is defined for all n, for all $A \in \mathcal{P}_n$, all $A_j \in \mathcal{P}_{n+1}$ with $A_j \subset A$.

We use these single-period risk neutral probabilities to define a probability measure Q on Ω. We do this inductively: for $n = 0, \ldots, N$, we define functions $Q_n : \mathcal{P}_n \mapsto (0, 1]$ by

$$Q_0(\Omega) = 1$$

and then if $A \in \mathcal{P}_{n+1}$, we define

$$Q_{n+1}(A) = q(A, B)Q_n(B),$$

where B is the unique set in $\mathcal{P}_n$ such that $A \subset B$. Then we take $Q = Q_N$, where if $A = \{\omega_1, \ldots, \omega_m\} \in \mathcal{P}_N$ consists of more than one point, we define $Q(\omega_i) > 0$ in such a way that $\sum_{i=1}^m Q(\omega_i) = Q(A)$.

It is easy to verify that Q so defined satisfies $Q(\omega) > 0$ for all ω, that $\sum_{\omega \in \Omega} Q(\omega) = 1$ and that the conditional probability $Q(A_j|A) = q(A_j, A)$ when $A \in \mathcal{P}_n$ and $A_j \subset A$ is in $\mathcal{P}_{n+1}$. (Hint: consider the functions Q_n and use induction on n. In particular, show by induction on n that if $A \in \mathcal{P}_n$, then $Q(A) = Q_n(A)$.)

Next it is easy to show that $\{\tilde{S}_n\}_{0 \le n \le N}$ is a $(Q, \{\mathcal{F}_n\}_{0 \le n \le N})$-martingale. This completes the proof of the theorem.

Valuation of Contingent Claims

Definition. A *contingent claim* C_N is an $\mathcal{F}_N$ measurable function. The contingent claim is said to be *attainable* if there is a self-financing portfolio with value process $\{V_n\}_{0 \le n \le N}$ satisfying $V_N = C_N$.

Theorem. *Suppose our multiperiod market is arbitrage-free so that there exists at least one martingale measure Q. Then the time zero price of an attainable European claim C_N is unique and is given by*

$$B_N^{-1} E^Q(C_N).$$

Remark. If the contingent claim is not priced this way, there is an arbitrage opportunity.

Proof of Theorem. There is a self-financing portfolio with value process V_n satisfying $V_N = C_N$. This portfolio may not be unique but V_0 must be the same for all such portfolios; otherwise there would be an arbitrage opportunity. Then the price of the claim must be V_0; otherwise there would be an arbitrage opportunity. Let Q be a martingale measure. Then by the proposition above, the discounted value process $\tilde{V}_n$ is a Q-martingale. Thus, for any martingale measure Q,

$$V_0 = \tilde{V}_0 = E^Q(\tilde{V}_N) = B_N^{-1} E^Q(V_N) = B_N^{-1} E^Q(C_N).$$

So the theorem is proved.

Completeness

Definition. Our multi-period market is said to be *complete* if all contingent claims are attainable.

Theorem. *Suppose our multiperiod market is arbitrage-free so that there exists at least one martingale measure. Then a contingent claim* C_N *is attainable if and only if* $E^Q(C_N)$ *does not depend on* Q*.*

Proof. Suppose first that C_N is attainable. Then we know from the previous theorem that $E^Q(C_N)$ does not depend on Q.

Conversely, we prove by induction on N, that if C_N is a contingent claim such that $E^Q(C_N)$ does not depend on Q, then it is attainable. We know it is true for $N = 1$. For $N \geq 2$, assuming it is true for $N-1$, we prove it is true for N. So, for $N \geq 2$, let C_N be a contingent claim such that $E^Q(C_N)$ does not depend on Q. We show C_N is attainable.

First let $A \in \mathcal{P}_{N-1}$ and suppose A is partitioned into the sets A_j, $j = 1, \ldots, m$ in $\mathcal{P}_N$. Let q_j be a martingale measure in the corresponding one-period model.

It is easy to show that

$$\sum_{j=1}^{m} q_j C_N(\omega_j) = E^Q(C_N|\mathcal{F}_{N-1})(\omega_j), (\omega_j \in A_j)$$

is independent of the martingale measure Q.

Since the theorem holds for $N = 1$, it follows that the claim C_N is attainable in each such one period model.

So there is a portfolio (ϕ_N, ψ_N), which is $\mathcal{F}_{N-1}$-measurable, such that

$$\phi_N S_N + \psi_N B_N = C_N.$$

Moreover

$$\phi_N S_{N-1} + \psi_N B_{N-1} = B_N^{-1} B_{N-1} E^Q(C_N | \mathcal{F}_{N-1})$$

for all martingale measures Q, with value independent of Q. We define this value to be C_{N-1}.

C_{N-1} is $\mathcal{F}_{N-1}$-measurable and has the property that

$$\begin{aligned} E^Q(C_{N-1}) &= B_N^{-1} B_{N-1} E^Q(E^Q(C_N | \mathcal{F}_{N-1})) \\ &= B_N^{-1} B_{N-1} E^Q(C_N) \end{aligned}$$

is independent of the martingale measure Q.

So, by the induction hypothesis, there is a self-financing portfolio in the initial $N-1$ period model with value process $\{V_n\}_{0\le n\le N-1}$ satisfying $V_{N-1} = C_{N-1} = \phi_N S_{N-1} + \psi_N B_{N-1}$.

So by extending this portfolio with (ϕ_N, ψ_N) we obtain a self-financing portfolio with $V_N = \phi_N S_N + \psi_N B_N = C_N$. Hence C_N is attainable and the proof is complete.

Theorem. *Suppose our multiperiod market is arbitrage-free so that there exists at least one martingale measure. Then the market is complete if and only if this martingale measure is unique (in the sense that if Q_1 and Q_2 are martingale measures, then $Q_1(A) = Q_2(A)$ for all $A \in \mathcal{F}_N$).*

Proof. If the martingale measure Q is unique, then it is trivial that $E^Q(C_N)$ is independent of Q for all contingent claims C_N. Then it follows from the previous theorem that C_N is attainable. Hence the market is complete.

Suppose now that the market is complete but there exist two distinct martingale measures Q_1 and Q_2. We will construct a contingent claim C_N such that $E^{Q_1}(C_N) \neq E^{Q_2}(C_N)$.

To this end, let n be the smallest n such that there is a one-period model with initial time n and terminal time $n+1$ such that the conditional probabilities corresponding to Q_1 and Q_2 differ.

Choose one of these one-period models and suppose it corresponds to $A \in \mathcal{P}_n$ and suppose A is partitioned into the sets A_j $(j = 1, \ldots, m)$ in $\mathcal{P}_{n+1}$.

For $j = 1, \ldots, m$, define

$$p_j = Q_1(A_j|A), \quad q_j = Q_2(A_j|A).$$

Since, by assumption, these risk-neutral probabilities differ and since we know the theorem holds in one-period models, there is a contingent claim C_j $(j = 1, \ldots, m)$ such that

$$c_1 = \sum_{j=1}^{m} p_j C_j \neq c_2 = \sum_{j=1}^{m} q_j C_j.$$

Then in the N-period model, we define the contingent claim

$$\begin{array}{ll} C_N(\omega) = C_j & (\omega \in A_j,\ j = 1, \ldots, m), \\ C_N(\omega) = 0 & (\omega \notin A). \end{array}$$

Now we show that $E^{Q_1}(C_N) \neq E^{Q_2}(C_N)$. Note if $i = 1, 2$ and $\omega \in A_j$,

$$E^{Q_i}(C_N|\mathcal{F}_{n+1})(\omega) = \sum sQ_i(C_N = s|A_j) = C_j.$$

Then if $\omega \in A$,

$$\begin{aligned} E^{Q_i}(C_N|\mathcal{F}_n)(\omega) &= E^{Q_i}(E^{Q_i}(C_N|\mathcal{F}_{n+1})|\mathcal{F}_n)(\omega) \\ &= \sum sQ_i(E^{Q_i}(C_N|\mathcal{F}_{n+1}) = s|A) \\ &= \sum_{j=1}^{m} C_j Q_i(A_j|A) \\ &= c_i. \end{aligned}$$

Clearly if $i = 1, 2$ and $\omega \notin A$,

$$E^{Q_i}(C_N|\mathcal{F}_n)(\omega) = 0.$$

Then for $i = 1, 2$,

$$E^{Q_i}(C_N) = E^{Q_i}(E^{Q_i}(C_N|\mathcal{F}_n)) = c_i Q_i(A)$$

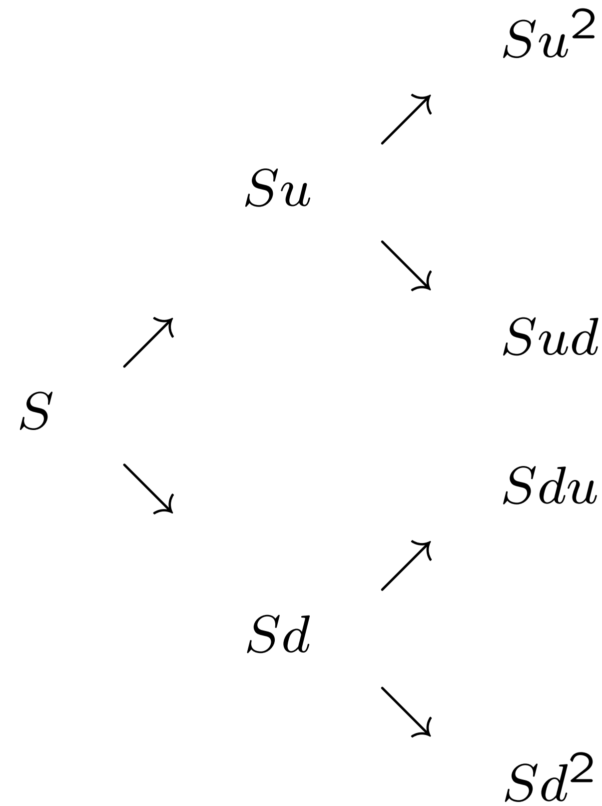
which are not equal, since $c_1 \neq c_2$ but $Q_1(A) = Q_2(A) > 0$. This completes the proof of the theorem.

A Complete Market: the N-Period Binomial Model

Consider the N-period binomial model with parameters u, d, $B_n = e^{rn\delta t}$ where $0 < d < u$. The sample space Ω consists of all the ordered N-tuples $\omega = (\omega^{(1)}, \ldots, \omega^{(N)})$, with $\omega^{(n)} = u$ or d and if $\omega = (\omega^{(1)}, \ldots, \omega^{(N)})$

$$S_n(\omega) = S_0 \omega^{(1)} \ldots \omega^{(n)}.$$

The case $N = 2$ is shown below:



The filtration is the natural filtration $\{\mathcal{F}_n\}_{0 \leq n \leq N}$ associated with the stock price process $\{S_n\}_{0 \leq n \leq N}$. In particular, $\mathcal{F}_N$ is the set of all subsets of Ω. Each A in $\mathcal{P}_n$ corresponds to an n-tuple $(\omega^{(1)}, \ldots, \omega^{(n)})$ so that if $\omega \in A$,

$$S_k(\omega) = S_0 \omega^{(1)}, \ldots, \omega^{(k)}$$

for $k = 1, \ldots, n$. Then in $\mathcal{P}_{n+1}$, A is partitioned into A_u and A_d, where A_u corresponds to $(\omega^{(1)}, \ldots, \omega^{(n)}, u)$ and A_d corresponds to $(\omega^{(1)}, \ldots, \omega^{(n)}, d)$.

We assume that

$$d < e^{r\delta t} < u.$$

The *martingale measure* Q is defined in the following way: let

$$p = \frac{e^{r\delta t} - d}{u - d}.$$

Then if the path ω has k up jumps and $N - k$ down jumps, we define

$$Q(\omega) = p^k (1 - p)^{N-k}.$$

We see that if $A \in \mathcal{P}_n$

$$Q(A_u|A) = p, \quad Q(A_d|A) = 1 - p.$$

We show that *the discounted stock price process* $\{\tilde{S}_n = B_n^{-1} S_n\}_{0 \le n \le N}$ *is a* $(Q, \{\mathcal{F}_n\}_{0 \le n \le N})$-*martingale*. If $\omega \in A$, a set in the partition $\mathcal{P}_n$ corresponding to $\mathcal{F}_n$, then if we write $S_n(\omega) = s$

$$\begin{aligned}
E(\tilde{S}_{n+1}|\mathcal{F}_n)(\omega) &= B_{n+1}^{-1} E(S_{n+1}|\mathcal{F}_n)(\omega) \\
&= B_{n+1}^{-1} \textstyle\sum_s s Q(S_{n+1} = s|A) \\
&= B_{n+1}^{-1} [suQ(A_u|A) + sdQ(A_d|A)] \\
&= B_{n+1}^{-1} [pS_n(\omega)u + (1-p)S_n(\omega)d] \\
&= B_{n+1}^{-1} S_n(\omega)[pu + (1-p)d] \\
&= B_{n+1}^{-1} S_n(\omega) e^{r\delta t} \\
&= B_n^{-1} S_n(\omega) \\
&= \tilde{S}_n(\omega).
\end{aligned}$$

A *contingent claim* is simply an $\mathcal{F}_N$-measurable random variable C_N, hence any function defined on Ω. To show the claim is attainable, we construct a replicating portfolio for it. To this end, we define the adapted process $\{C_n\}_{0\le n\le N}$ by

$$C_n = e^{-r(N-n)\delta t} E^Q(C_N|\mathcal{F}_n).$$

Then the discounted process $\{\tilde{C}_n = B_n^{-1}C_n\}_{0\le n\le N}$ is a martingale since

$$\tilde{C}_n = E^Q(\tilde{C}_N|\mathcal{F}_n).$$

Note that if $\omega \in A \in \mathcal{P}_{n-1}$,

$$\begin{aligned} \tilde{C}_{n-1}(\omega) &= E^Q(\tilde{C}_n|\mathcal{F}_{n-1})(\omega) \\ &= \sum sQ(\tilde{C}_n = s|A) \\ &= B_n^{-1}[C_n(u)Q(A_u|A) + C_n(d)Q(A_d|A)] \end{aligned}$$

so that

$$C_{n-1}(\omega) = e^{-r\delta t}[pC_n(u) + (1-p)C_n(d)].$$

Then we construct the *self-financing replicating portfolio* $\{(\phi_n, \psi_n)\}_{1 \le n \le N}$ for the claim C_N as follows: if $\omega \in A$, a set in the partition $\mathcal{P}_{n-1}$ corresponding to $\mathcal{F}_{n-1}$, then we define

$$\phi_n(\omega) = \frac{C_n(u) - C_n(d)}{S_{n-1}(\omega)(u-d)},$$
$$\psi_n(\omega) = [C_{n-1}(\omega) - \phi_n(\omega)S_{n-1}(\omega)]B_{n-1}^{-1},$$

where $C_n(u)$ is the value of $C_n(\omega)$ on the set

$$A_u = \{\omega \in A : S_n(\omega) = S_{n-1}(\omega)u\}$$

and $C_n(d)$ is the value of $C_n(\omega)$ on the set

$$A_d = \{\omega \in A : S_n(\omega) = S_{n-1}(\omega)d\}.$$

Note that with this definition for all ω and $n = 1, \ldots, N$

$$\phi_n S_{n-1} + \psi_n B_{n-1} = C_{n-1}. \qquad (1)$$

Also if $\omega \in A_u$ (and similarly if $\omega \in A_d$),

$$
\begin{aligned}
&\phi_n(\omega)S_n(\omega) + \psi_n(\omega)B_n \\
&= \phi_n(\omega)S_{n-1}(\omega)u + [C_{n-1}(\omega) - \phi_n(\omega)S_{n-1}(\omega)]e^{r\delta t} \\
&= \phi_n(\omega)S_{n-1}(\omega)(u - e^{r\delta t}) + C_{n-1}(\omega)e^{r\delta t} \\
&= \tfrac{(C_n(u)-C_n(d))(u-e^{r\delta t})}{u-d} + pC_n(u) + (1-p)C_n(d) \\
&= (C_n(u) - C_n(d))(1-p) + pC_n(u) + (1-p)C_n(d) \\
&= C_n(u) \\
&= C_n(\omega).
\end{aligned}
$$

So for $n = 1, \ldots, N$

$$\phi_n S_n + \psi_n B_n = C_n. \qquad (2)$$

In particular (2) shows that

$$\phi_N S_N + \psi_N B_N = C_N$$

so that the portfolio does replicate the claim and together with (1) it shows that C_n is the value process of the portfolio.

However, (1) and (2) together also show that the portfolio is self-financing. By arbitrage arguments, it follows that $\{C_n\}_{0 \leq n \leq N}$ is the *value process* for the claim.

Also we have shown that every claim is attainable so that the market is complete and the martingale measure is unique.