Valuation of options in the Black-Scholes model

First we review the Black-Scholes pricing of vanilla calls and puts. For barriers we use Buchen's PDE method to derive the Black-Scholes price. To price Asians, we derive a partial differential equation with one state variable. We also derive a tight lower bound for the price of such an option.

Valuation of vanilla calls and puts

The Black-Scholes market consists of two securities. The first is the cash bond $B_t = e^{rt}$, where r is the risk-free rate. The second is the risky asset (usually a stock) which we denote by S_t. We suppose that S_t is a geometric Brownian motion so that

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

for some constants μ and σ, where W_t is a P-Brownian motion. We apply Ito's formula to get

$$d(\log S_t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t.$$

Hence

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}.$$

In the risk-neutral world, the probability P is replaced by Q and with respect to Q, the drift μ is replaced by the interest rate r so that

$$S_t = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t}.$$

Q is a martingale measure, that is, with respect to it the process $e^{-rt}S_t$ is a martingale.

Consider a contingent claim C_T. Its price at time 0 is given by

$$C_0 = e^{-rT} E_Q(C_T).$$

Suppose the claim $C_T = f(S_T)$ depends only on S_T. We can value by one of two methods:

Method 1: Directly from the risk-neutral formula to get the price at $t = 0$

$$C_0 = e^{-rT} \int_{-\infty}^{\infty} f\left(S_0 e^{(r-\sigma^2/2)T + \sigma y \sqrt{T}}\right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy.$$

In particular, the fair price of the call with strike K and maturity T at time $t = 0$ is given by the famous Black-Scholes formula

$$C_0 = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2),$$

where

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}, \quad \Phi(d)$$

We can use put-call parity $S_0 + P_0 = C_0 + K e^{-rT}$ to get the put price

$$P_0 = -S_0 \Phi(-d_1) + K e^{-rT} \Phi(-d_2).$$

Method 2: Otherwise we can solve a PDE. Since the claim $C_T = f(S_T)$ depends only on S_T, its value at time t is given by a function $V(t, S_t)$. By Ito's formula

$$\begin{aligned} dV &= V_t dt + V_x dS_t + \tfrac{1}{2} V_{xx} d[S]_t \\ &= V_t dt + V_x (r S_t dt + \sigma S_t dW_t) + \tfrac{1}{2} V_{xx} \sigma^2 S_t^2 dt \\ &= \left[V_t + V_x r S_t + \tfrac{1}{2} V_{xx} \sigma^2 S_t^2 \right] dt + V_x \sigma S_t dW_t. \end{aligned}$$

$V(t, S_t)$ is also the value process of a self-financing portfolio and hence $\tilde{V}_t = e^{-rt} V(t, S_t)$ is also a $Q-$martingale. By Ito's Lemma again

$$\begin{aligned} d\tilde{V}_t &= -r e^{-rt} V_t dt + e^{-rt} dV_t \\ &= e^{-rt} \left[-rV + V_t + V_x r S_t + \tfrac{1}{2} V_{xx} \sigma^2 S_t^2 \right] dt \\ &\quad + e^{-rt} V_x \sigma S_t dW_t. \end{aligned}$$

By the martingale property, the coefficient of dt is 0 and so $V(t, x)$ solves the Black-Scholes PDE

$$-rV + V_t + rxV_x + \frac{1}{2} \sigma^2 x^2 V_{xx} = 0$$

subject to

$$V(T, x) = f(x).$$

This can be transformed to the Heat Equation.

Valuation of barrier option prices using the risk-neutral formula

Eight Types of Barrier Options

down-and-out, down-and-in

up-and-out, up-and-in

calls and puts

For an up-and-out call with barrier B, strike K and maturity T, the payoff is

$$V_T = \max\{S_T - K, 0\}$$

if $S_t \leq B$ for $0 \leq t \leq T$, otherwise it is 0.

For a down-and-in put with barrier B, strike K and maturity T, the payoff is

$$V_T = \max\{K - S_T, 0\}$$

if $S_t < B$ for some t in $0 \leq t \leq T$, otherwise it is 0.

Up-and-out Call

We can use the risk-neutral formula to show if $K < B$ and $S_0 < B$, the price at time $t = 0$ of the barrier option is

$$S_0\left[\Phi\left(\delta_+\left(T,\frac{S_0}{K}\right)\right) - \Phi\left(\delta_+\left(T,\frac{S_0}{B}\right)\right)\right]$$

$$-Ke^{-rT}\left[\Phi\left(\delta_-\left(T,\frac{S_0}{K}\right)\right) - \Phi\left(\delta_-\left(T,\frac{S_0}{B}\right)\right)\right]$$

$$-S_0\left(\frac{S_0}{B}\right)^{-2r/\sigma^2-1}\Big[\Phi\Big(\delta_+\Big(T,\frac{B^2}{KS_0}\Big)\Big) - \Phi\Big(\delta_+\Big(T,\frac{B}{S_0}\Big)$$

$$+Ke^{-rT}\left(\frac{S_0}{B}\right)^{-2r/\sigma^2+1} *$$
$$*\left[\Phi\left(\delta_-\left(T,\frac{B^2}{KS_0}\right)\right) - \Phi\left(\delta_-\left(T,\frac{B}{S_0}\right)\right)\right],$$

where

$$\delta_\pm(\tau, s) = \frac{1}{\sigma\sqrt{\tau}}[\log\ s + (r \pm \sigma^2/2)\tau].$$

Using PDEs and the method of images to price barriers

Suppose $S_0 < B$, $B < K$. Then as long as the barrier has not been hit, the value of an up and out call is given by a function $v(t, S_t)$, where $v(t,x)$ solves the Black-Scholes PDE

$$v_t + rxv_x + \frac{1}{2}\sigma^2 x^2 v_{xx} - rv = 0.$$

Also v satisfies the conditions

$$v(t,0) = 0, \quad v(t,B) = 0,$$

$$v(T,x) = \max(x-K,0) = (x-K)^+,\ 0 \le x \le B.$$

We make the transformation $\tau = T - t$ and then these become

$$v_\tau - rxv_x - \frac{1}{2}\sigma^2 x^2 v_{xx} + rv = 0. \qquad (1)$$

$$v(\tau,0) = 0, \quad v(\tau,B) = 0,$$

$$v(0,x) = \max(x-K,0) = (x-K)^+,\ 0 \le x \le B.$$

Lemma. If $v(\tau, x)$ solves (1), then so also does

$$v^*(\tau, x) = \left(\frac{B}{x}\right)^{\alpha} v\left(\tau, \frac{B^2}{x}\right), \quad \alpha = \frac{2r}{\sigma^2} - 1.$$

Proof. Direct substitution.

Now we define

$$U_\xi(\tau, x) = x\Phi(d_1), \quad V_\xi(\tau, x) = e^{-r\tau}\Phi(d_2),$$

where

$$d_1 = \frac{\log(x/\xi) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}.$$

U_ξ and V_ξ are solutions of (1) since the second is the value of a digital call with strike ξ and maturity τ when the stock price is x, and the first is the value of a call and ξ digital calls with strike ξ and maturity τ when the stock price is x.

Next we define

$$C_\xi(\tau,x) = U_\xi(\tau,x) - KV_\xi(\tau,x), \quad C^*_\xi(\tau,x) = U^*_\xi(\tau,x) -$$

These are also solutions of (1) and we find that the price of an up and out call is

$$C_K(\tau,S_\tau) - C^*_K(\tau,S_\tau) - [C_B(\tau,S_\tau) - C^*_B(\tau,S_\tau)].$$

References

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Buchen, P., Pricing European barrier options, *University of Sydney Research Report*, 1996.

Asian options: a PDE with one state variable

Asian Option with Floating Strike

The payoff functions are given by, respectively for the Asian call and the Asian put:

$$AC_{\text{float}} = \max\left\{S_T - \frac{1}{T}\int_0^T S_u du, 0\right\},$$

$$AP_{\text{float}} = \max\left\{\frac{1}{T}\int_0^T S_u du - S_T, 0\right\}.$$

Asian Option With Fixed Strike

The payoff functions are given by, respectively for the Asian call and the Asian put:

$$AC_{\text{fixed}} = \max\left\{\frac{1}{T}\int_0^T S_u du - K, 0\right\},$$

$$AP_{\text{fixed}} = \max\left\{K - \frac{1}{T}\int_0^T S_u du, 0\right\}.$$

Asian call with fixed strike

Write $Y_t = \int_0^t S_u du$ and define

$$V_t = e^{-r(T-t)} E_Q \left[\frac{1}{T} Y_T - K | \mathcal{F}_t \right].$$

Since (S_t, Y_t) has the Markov property, there exists a function $v(t, x, y)$ such that

$$V_t = v(t, S_t, Y_t)$$

and we find that $v(t, x, y)$ solves the PDE.

To value the option, we could solve this PDE which has two state variables.

Instead we derive another PDE with just one state variable. We suppose $r > 0$.

Write

$$X_T = \frac{1}{T}\int_0^T S_t dt - K.$$

Then we find that

$$X_t = e^{-r(T-t)}E_Q[X_T|\mathcal{F}_t] = \gamma_t S_t + Z_t e^{rt} - e^{-r(T-t)}K,$$

where

$$\gamma_t = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right), \qquad Z_t = \int_0^t S_u du.$$

This means that X_t is the value at time t of a self-financing portfolio consisting of γ_t shares and $Z_t - Ke^{-rT}$ bonds.

X_t satisfies

$$dX_t = r(X_t - \gamma_t S_t)dt + \gamma_t dS_t$$

and

$$X_T = \frac{1}{T}\int_0^T S_t dt - K.$$

So

$$V_T = X_T^+ = \max\{X_T, 0\}.$$

Now define

$$Y_t = \frac{X_t}{S_t}.$$

Then by Ito's formula

$$dY_t = \sigma[\gamma_t - Y_t][dW_t - \sigma dt].$$

We use Girsanov's theorem to change to a new measure $\mathcal{R}$ so that

$$\tilde{W}_t = W_t - \sigma t$$

becomes a Brownian motion. $\mathcal{R}$ is defined by

$$\mathcal{R}(A) = E_Q[L_T I_A],$$

where

$$L_T = e^{\sigma W_T - \sigma^2 T/2} = \frac{e^{-rT} S_T}{S_0}.$$

Now

$$dY_t = \sigma[\gamma_t - Y_t] d\tilde{W}_t$$

so that Y_t is an $\mathcal{R}-$martingale and we find that

$$V_t = S_t E_{\mathcal{R}}[Y_T^+ | \mathcal{F}_t].$$

Since Y_t is Markov, there is a function $g(t, y)$ such that

$$g(t, Y_t) = E_{\mathcal{R}}\left[Y_T^+ | \mathcal{F}_t\right].$$

By Ito's lemma,

$$dg(t, Y_t) = \left[g_t + \frac{1}{2}\sigma^2(\gamma_t - Y_t)^2 g_{yy}\right] dt + \sigma(\gamma_t - Y_t) g_y d\tilde{W}_t.$$

Since $g(t, Y_t)$ is an $\mathcal{R}-$martingale, the dt term must be zero and so g solves the PDE

$$g_t + \frac{1}{2}\sigma^2(\gamma_t - y)^2 g_{yy} = 0, \quad 0 \le t < T,\ y \in R$$

with boundary conditions for $y \in R$, $0 \le t \le T$

$$g(T, y) = y^+, \quad \lim_{y \to -\infty} g(t, y) = 0, \quad \lim_{y \to \infty} [g(t, y) - y] =$$

Then the value of the option at time t is

$$V_t = S_t g\left(t, \frac{\frac{1}{T}\int_0^T S_t dt - K}{S_t}\right).$$

Asian Options: a Lower Bound

The time $t = 0$ price of an Asian call with fixed strike K is

$$e^{-rT} E_Q[Y^+], \quad \text{where} \quad Y = \frac{1}{T}\int_0^T S_t \, dt - K.$$

Writing

$$Z = \int_0^T W_t \, dt,$$

we use Jensen's inequality to get

$$E_Q[Y^+] = E[E[Y^+|Z]] \geq E[E[Y|Z]^+].$$

Note that this Z is essentially the geometric average of the stock price since in the risk-neutral world

$$\frac{1}{T}\int_0^T \log S_t dt = \frac{(r - \sigma^2/2)T}{2} + \frac{\sigma}{T} Z.$$

If g is the density function of Z,

$$E(E(Y|Z)^+) = \int_{-\infty}^{\infty} E(Y|Z = z)^+ g(z) dz.$$

Calculation of $E(Y|Z = z)$:

Z is normally distributed with mean 0 and variance

$$E(Z^2) = \int_0^T \int_0^T E(W_t W_s) ds dt = T^3/3.$$

For fixed t, W_t and Z have a bivariate normal distribution with

$$\text{Cov}(W_t, Z) = \int_0^T \text{Cov}(W_t, W_u) du = Tt - t^2/2.$$

Hence the conditional distribution of W_t given $Z = z$ is normal with mean

$$\frac{3t(T - t/2)}{T^3} z$$

and variance

$$\left(1 - \frac{3t(T - t/2)^2}{T^3}\right) t.$$

Then from the MGF

$$E(\exp(\sigma W_t)|Z = z) = e^{R(t)},$$

where

$$R(t) = \sigma\frac{3t(T - t/2)}{T^3}z+\frac{1}{2}\sigma^2\left(1 - \frac{3t(T - t/2)^2}{T^3}\right)t.$$

So

$$E\left(\frac{1}{T}\int_0^T S_t\,dt|Z = z\right)$$

$$= \frac{1}{T}\int_0^T E(\exp(\ln(S_0) + (r - \sigma^2/2)t + \sigma W_t)|Z = z)$$

$$= \frac{S_0}{T}\int_0^T \exp((r - \sigma^2/2)t)E(\exp(\sigma W_t)|Z = z)dt$$

$$= \frac{S_0}{T}\int_0^T \exp((r - \sigma^2/2)t + R(t))dt$$

So $E(Y|Z = z) = \frac{S_0}{T}\int_0^T e^{Q(t)}dt - K$,
where $Q(t) = (r - \sigma^2/2)t + R(t)$.

When is $E(Y|Z=z)$ positive?

By convexity,

$$
\begin{aligned}
\frac{1}{T}\int_0^T S_t\,dt &= \frac{1}{T}\int_0^T \exp(\ln(S_t))dt \\
&\geq \exp\left(\frac{1}{T}\int_0^T \ln(S_t)dt\right) \\
&= \exp\Bigg(\frac{1}{T}\int_0^T [\ln(S_0) + (r-\sigma^2/2)t + \sigma W_t \\
&= S_0 \exp((r-\sigma^2/2)T/2 + \sigma Z/T) \\
&\geq K,
\end{aligned}
$$

if

$$Z \geq k = T(\ln(K/S_0) - (r-\sigma^2/2)T/2)/\sigma.$$

So

$$E(Y|Z=z) \geq 0 \quad \text{if} \quad z \geq k.$$

On the other hand,

$$
\begin{aligned}
Q(t) &= \left(r - \frac{\sigma^2}{2}\right)t + \sigma\frac{3t(T-t/2)}{T^3}z + \frac{\sigma^2}{2}\left(1 - \frac{3t(T-t/2)^2}{T^3}\right. \\
&\le \max\{0, (r-\sigma^2/2)T\} + \frac{3\sigma t}{T^2}z + \frac{1}{2}\sigma^2 T \\
&= C + \alpha t z,
\end{aligned}
$$

where

$$C = \max\{r, \sigma^2 T/2\}, \quad \alpha = \frac{3\sigma}{T^2}.$$

So when $z < 0$,

$$\frac{1}{T}\int_0^T e^{Q(t)}dt \le \frac{e^C}{T}\int_0^T e^{\alpha t z}dt = \frac{e^C}{T}\frac{e^{\alpha T z} - 1}{\alpha z} \le \frac{e^C}{-T\alpha z}.$$

Hence

$$\frac{S_0}{T}\int_0^T e^{Q(t)}dt \le K \text{ if } z \le -\frac{e^C S_0 T}{3\sigma K}.$$

So

$$E(Y|Z=z) \le 0 \quad \text{if} \quad z \le -\frac{e^C S_0 T}{3\sigma K}.$$

Now we can see from differentiating the integral expression for $E(Y|Z = z)$ with respect to z that the derivative is positive.

So there is a unique z_0 such that

$$E(Y|Z = z_0) = 0.$$

Clearly

$$-\frac{e^C S_0 T}{3\sigma K} \leq z_0 \leq k.$$

The lower bound:

Then the lower bound is

$$S_0 I_1 - K I_2,$$

where

$$I_1 = \frac{\sqrt{3}}{T^{3/2}} \frac{1}{T} \int_0^T e^{\left(r - \frac{3\sigma^2 t (T - t/2)^2}{2T^3}\right)t} \frac{1}{\sqrt{2\pi}} \left[\int_{z_0}^{\infty} e^{g(t,z)} dz \right] dt$$

$$g(t,z) = \sigma \frac{3t(T-t/2)}{T^3} z - 3z^2/(2T^3)$$

$$I_2 = \frac{\sqrt{3}}{T^{3/2}} \frac{1}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{-3z^2/(2T^3)} dz.$$

Note that

$$\begin{aligned} g(t,z) &= -\tfrac{3}{2T^3}(z^2 - 2\sigma t(T - t/2)z) \\ &= -\tfrac{3}{2T^3}[(z - \sigma t(T - t/2))^2 - \sigma^2 t^2 (T - t/2)^2] \\ &= -\tfrac{3}{2T^3}(z - \sigma t(T - t/2))^2 + \tfrac{3}{2T^3}\sigma^2 t^2 (T - t/2 \end{aligned}$$

So the integrand of I_1 can be written as

$$e^{rt}\frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{-\frac{3}{2T^3}(z-\sigma t(T-t/2))^2)}dz$$

$$= e^{rt}\frac{1}{\sqrt{2\pi}}\int_{z_0-\sigma t(T-t/2)}^{\infty} e^{-\frac{3}{2T^3}u^2}du$$

so that

$$I_1 = \frac{1}{T}\int_0^T e^{rt}\Phi(-\sqrt{3}(z_0 - \sigma t(T-t/2))/T^{3/2})dt,$$

where Φ is the standard normal cdf. Also

$$I_2 = \Phi(-\sqrt{3}z_0/T^{3/2}).$$

Hence, after discounting, the lower bound for the option price is

$$\frac{S_0}{T}\int_0^T e^{-r(T-t)}\Phi\left(\frac{-\sqrt{3}z_0 + \sigma\sqrt{3}t(T-t/2)}{T^{3/2}}\right)dt$$

$$-Ke^{-rT}\Phi\left(-\frac{\sqrt{3}z_0}{T^{3/2}}\right).$$

This lower bound is often very close to the price of the option.

Valuation of exotic options in the binomial model

We study the valuation of barrier, lookback and Asian options in the binomial model.

For barriers we use the usual binomial tree but for lookbacks we use the Cheuk-Vorst lattice and for Asians we use the Hull-White method as modified by Dai, Huang and Lyuu.

We also consider the convergence to the Black-Scholes price.

Convergence of binomial price to Black-Scholes price

In the continuous framework, we model the stock price S_t using *Geometric Brownian Motion*,

$$dS_t = \nu S_t dt + \sigma S_t dW_t,$$

for which the two parameters are the “drift” ν and the “volatility” σ.

However, in valuing an option with maturity T at time $t = 0$, the drift is replaced by the interest rate r (this corresponds to replacing the “real-world” probability by the risk-neutral probability in the binomial model).

The result is that in the risk-neutral world, the distribution of $\log(S_T)$ is normal with mean $\log(S_0) + (r - \sigma^2/2)T$ and variance $\sigma^2 T$.

We can approximate this by the N-period binomial model by taking

$$\delta t = T/N, \quad u = e^{\sigma\sqrt{\delta t}}, \quad d = e^{-\sigma\sqrt{\delta t}}.$$

Then at time T the stock price will be

$$S_T = S_0 u^{X_N} d^{N-X_N} = S_0 e^{\sigma\sqrt{T}\left(\frac{2X_N - N}{\sqrt{N}}\right)},$$

where X_N is the number of "up jumps".

Under the martingale measure Q, the probability of an up jump is

$$q = \frac{e^{r\delta t} - d}{u - d} = \frac{e^{r\delta t} - e^{-\sigma\sqrt{\delta t}}}{e^{\sigma\sqrt{\delta t}} - e^{-\sigma\sqrt{\delta t}}}$$

$$= \frac{1}{2} + \frac{r - \frac{1}{2}\sigma^2}{2\sigma}\sqrt{\delta t} + O(\delta t),$$

where r is the risk-free rate. Note that q depends on N and approaches the limit 1/2 as $N \to \infty$.

To find the limiting distribution of S_T as $N \to \infty$, we use:

Theorem. (Central Limit Theorem) *For $N \geq 1$, let ξ_1^N, ξ_2^N, ..., ξ_N^N be independent identically distributed random variables with mean μ_N and finite nonzero variance σ_N^2 satisfying*

$$N\mu_N \to \mu, \quad N\sigma_N^2 \to \sigma^2$$

as $N \to \infty$.

Then $Y_N = \xi_1^N + \xi_2^N + \cdots + \xi_N^N$ converges in distribution to an $N(\mu, \sigma^2)$ random variable Z as $N \to \infty$, that is, for all y

$$P(Y_N \leq y) \to P(Z \leq y) \quad \text{as} \quad N \to \infty.$$

Note that

$$X_N = \sum_{n=1}^{N} \xi_n,$$

where ξ_1, ξ_2,....,ξ_N are i.i.d random variables where $Q\{\xi_n = 1\} = q$, $Q\{\xi_n = 0\} = 1 - q$. Next

$E\xi_n = q$, $\text{Var}(\xi_n) = q(1-q)^2+(1-q)(-q)^2 = q(1-q)$.

If for $n = 1, ..., N$ we define

$$\xi_n^N = \frac{2\xi_n - 1}{\sqrt{N}},$$

then we see that

$$Y_N = (2X_N - N)/\sqrt{N} = \sum_{n=1}^{N} \xi_N^n,$$

where the N random variables $\{\xi_n^N\}_{1\leq n\leq N}$ are independent and take the value $1/\sqrt{N}$ with probability q and $-1/\sqrt{N}$ otherwise. This means

$$E(\xi_n^N) = \mu_N = (2q - 1)/\sqrt{N},$$

$$\text{Var}(\xi_n^N) = \sigma_N^2 = \frac{4}{N}q(1 - q).$$

So as $N \to \infty$,

$$N\mu_N = \sqrt{N}(2q-1) \to \sqrt{T}\left(\frac{r - \frac{1}{2}\sigma^2}{\sigma}\right), \quad N\sigma_N^2 \to 1.$$

So by the Central Limit Theorem, the distribution of the random variable $Y_N = (2X_N - N)/\sqrt{N}$ converges to normal with mean $\sqrt{T}\left(\frac{r-\frac{1}{2}\sigma^2}{\sigma}\right)$ and variance 1. Thus in the limit

$$\log S_T = \log S_0 + \sigma\sqrt{T}\left(\frac{2X_N - N}{\sqrt{N}}\right) = \log S_0 + \sigma\sqrt{T}Y_N$$

has a normal distribution with mean

$$\log S_0 + \sigma\sqrt{T}\sqrt{T}\left(\frac{r - \frac{1}{2}\sigma^2}{\sigma}\right) = \log S_0 + \left(r - \frac{1}{2}\sigma^2\right)T$$

and variance $\sigma^2 T$, or we can write

$$S_T = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Y},$$

where Y is $N(0,1)$. This is exactly the distribution obtained if we model the stock price using Geometric Brownian Motion with "drift" r and "volatility" σ.

Limit of the Binomial Call Price as $N \to \infty$:

Now in the N-period binomial model with parameters S_0, u, d, r, the price of a European call option with strike K and maturity T is

$$C_N = e^{-rT} \sum_{k=0}^{N} \binom{N}{k} q^k (1-q)^{N-k} \max\{S_0 u^k d^{N-k} - K, 0\}$$

where $q = (e^{r\delta t} - d)/(u - d)$. Then

$$C_N = e^{-rT} \sum_{k=j}^{N} \binom{N}{k} q^k (1-q)^{N-k} (S_0 u^k d^{N-k} - K),$$

where j is the smallest integer k such that $S_0 u^k d^{N-k} - K > 0$. Next we can write it as

$$C_N = S_0 \sum_{k=j}^{N} \binom{N}{k} (q')^k (1 - q')^{N-k}$$

$$-Ke^{-rT} \sum_{k=j}^{N} \binom{N}{k} q^k (1 - q)^{N-k},$$

where $q' = que^{-r\delta t}$.

We want to find the limit of C_N as $N \to \infty$ with

$$u = e^{\sigma\sqrt{\delta t}}, \quad d = e^{-\sigma\sqrt{\delta t}}.$$

First note that

$$\sum_{k=j}^{N} \binom{N}{k} q^k (1-q)^{N-k} = Q(X_N \geq j) = 1 - Q(X_N \leq j-1)$$

Now as seen above,

$$\frac{2X_N - N}{\sqrt{N}} - \mu \to Y$$

in distribution as $N \to \infty$, where Y is a $N(0,1)$ random variable and

$$\mu = \sqrt{T}(r - \sigma^2/2)/\sigma.$$

Next note that

$$Q(X_N \leq j-1)$$

$$= Q\left(\frac{2X_N - N}{\sqrt{N}} - \mu \leq \frac{2(j-1) - N}{\sqrt{N}} - \mu\right).$$

Now we know that

$$Q\left(\frac{2X_N - N}{\sqrt{N}} - \mu \leq y\right) \to \Phi(y)$$

as $N \to \infty$ for a fixed y, where Φ is the standard normal distribution function

$$\Phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx.$$

So we need to see what happens to

$$\frac{2(j-1) - N}{\sqrt{N}} - \mu$$

as $n \to \infty$.

From the definition of j:

$$-\frac{\log(S_0/K)}{\sigma\sqrt{T}} - \frac{2}{\sqrt{N}} < \frac{2(j-1) - N}{\sqrt{N}} \leq -\frac{\log(S_0/K)}{\sigma\sqrt{T}}$$

Hence

$$\frac{2(j-1) - N}{\sqrt{N}} - \mu \to -\frac{\log(S_0/K)}{\sigma\sqrt{T}} - \mu = -d_2$$

as $N \to \infty$, where

$$d_2 = \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

So, as $N \to \infty$,

$$\begin{aligned} & Q(X_N \le j-1) \\ & = Q\left(\frac{2X_N - N}{\sqrt{N}} - \mu \le \frac{2(j-1)-N}{\sqrt{N}} - \mu\right) \\ & \to \Phi(-d_2), \end{aligned}$$

by the Central Limit Theorem.

Hence as $N \to \infty$,

$$\sum_{k=j}^{N} \binom{N}{k} q^k (1-q)^{N-k} = 1 - Q(X_N \le j-1)$$

$$\to 1 - \Phi(-d_2) = \Phi(d_2).$$

Now we consider

$$\sum_{k=j}^{N} \binom{N}{k} (q')^k (1-q')^{N-k},$$

where $q' = que^{-r\delta t}$. Then

$$q' = \frac{e^{\sigma\sqrt{\delta t}} - e^{-r\delta t}}{e^{\sigma\sqrt{\delta t}} - e^{-\sigma\sqrt{\delta t}}} = \frac{1}{2} + \frac{r + \frac{1}{2}\sigma^2}{2\sigma}\sqrt{\delta t} + O(\delta t).$$

We follow the same reasoning as above but now d_2 is replaced by

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}.$$

We conclude that as $N \to \infty$,

$$\sum_{k=j}^{N} \binom{N}{k} (q')^k (1-q')^{N-k} \to 1 - \Phi(-d_1) = \Phi(d_1).$$

Hence as $N \to \infty$,

$$C_N \to S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2),$$

where

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

This is the *Black-Scholes formula*.

Barrier Options

Out-in Parity

A portfolio of an down-and-out call and a down-and-in-call is equivalent to a standard call.

Hence the price of an down-and-out call equals the price of a standard call minus the price of an down-and-in call.

Binomial pricing

It is easier to price an “out” option in the binomial model than an “in” option.

A Matlab program to calculate the binomial price of a down-and-out call:

```
function p = bindownoutcall(S,u,d,K,r,T,B,n)
p = (exp(r*T/n)-d)/(u-d);
for j = 1:n+1
if S*u∧(n-j+1)*d∧(j-1)> B
C(n+1,j) = max(S*u∧(n-j+1)*d∧(j-1)-K,0);
else
C(n+1,j) = 0;
end
end
for i = n:-1:1
for j = 1:i
if S*u∧(i-j+1)*d∧(j-1)> B
C(i,j) = exp(-r*T/n)*(p*C(i+1,j)+(1-p)*C(i+1,j+
else
C(i,j) = 0;
end
end
end
p=C(1,1);
```

Binomial approximation to Black-Scholes price

Let $C(n)$ be the binomial price calculated by the $n-$period binomial model with

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}.$$

For a vanilla (European) call option,

$$|C(n) - C_{BS}| = O\left(\frac{1}{n}\right),$$

where C_{BS} is the Black-Scholes price. However for a barrier option, we only have

$$|C(n) - C_{BS}| = O\left(\frac{1}{\sqrt{n}}\right),$$

In order to speed up the convergence of the binomial price of European barrier options as the number of periods n tends to infinity, Boyle and Lau (1994) selected n so that the barrier B was as close as possible to a stock price in the lattice.

In fact, in the case of down options, B is exactly a price in the lattice if and only if

$$n = F(m) = m^2\sigma^2 T/(\log(S_0/B))^2$$

for some positive integer m.

So for down options when $S_0 > B$, Boyle and Lau selected $n = n_m$ as the largest integer less than or equal to $F(m)$ for $m = 1, 2, 3, \ldots$.

Then we find that

$$|C(n_m) - C_{BS}| = O\left(\frac{1}{n_m}\right).$$

Lookback Options

Lookback Option with Floating Strike

The payoff functions are given by, respectively for the lookback call and the lookback put:

$$LC_{\text{float}} = S_T - \min_{0 \le t \le T} S_t, \quad LP_{\text{float}} = \max_{0 \le t \le T} S_t - S_T.$$

Lookback Option With Fixed Strike

The payoff functions are given by, respectively for the lookback call and the lookback put:

$$LC_{\text{fix}} = \max\left\{\max_{0 \le t \le T} S_t - K, 0\right\},$$

$$LP_{\text{fix}} = \max\left\{K - \min_{0 \le t \le T} S_t, 0\right\}.$$

where K is the strike price.

Lookback Put with Floating Strike

Using risk-neutral valuation (or from a PDE), the Black-Scholes price of the option can be derived as

$$
\begin{aligned}
& v(t, S_t, Y_t) \\
& = \left(1 + \frac{\sigma^2}{2r}\right) S_t \Phi(\delta_+(\tau, S_t/Y_t)) \\
& \qquad + e^{-r\tau} Y_t \Phi(-\delta_-(\tau, S_t/Y_t)) \\
& \qquad - \frac{\sigma^2}{2r} e^{-r\tau} (Y_t/S_t)^{2r/\sigma^2} S_t \Phi(-\delta_-(\tau, Y_t/S_t)) - S_t,
\end{aligned}
$$

where

$$
Y_t = \max_{0 \le u \le t} S_u, \qquad \tau = T - t,
$$

$$
\delta_\pm(\tau, d) = \frac{\ln(d) + (r \pm \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.
$$

Binomial pricing

We use the $n-$period binomial model with

$$u = e^{\sigma \Delta t}, \quad d = u^{-1}, \quad \Delta t = T/n.$$

The risk-neutral probability is

$$p = \frac{e^{r\Delta t} - d}{u - d}.$$

This is a path-dependent option and it is not convenient to use the usual binomial tree to price it. Instead we introduce another *state variable*.

Denote by S_i the stock price at time $t_i = i\Delta t$ and set

$$Y_i = \max_{0 \le j \le i} S_j.$$

Next set

$$Z_i = Y_i / S_i.$$

The possible values for Z_i are u^m, where $0 \leq m \leq i$.

If $Z_i = 1$, then $Z_{i+1} = 1$ with probability p and equals u with probability $1 - p$.

If $Z_i = u^j$ with $j > 0$, then $Z_{i+1} = u^{j-1}$ with probability p and equals u^{j+1} with probability $1 - p$.

The payoff to the option is $Y_n - S_n = S_n(Z_n - 1)$.

Let $S_i V_{ij}$ be the value of the option at time t_i when the stock price is S_i and $Z_i = u^j$.

Then by risk-neutral valuation, when $j \geq 1$,

$$S_i V_{ij} = e^{-r\Delta t}(pS_i u V_{i+1,j-1} + (1-p)S_i d V_{i+1,j+1})$$

and when $j = 0$

$$S_i V_{i0} = e^{-r\Delta t}(pS_i u V_{i+1,0} + (1-p)S_i d V_{i+1,1}).$$

Thus when $j \geq 1$,

$$V_{ij} = e^{-r\Delta t}(puV_{i+1,j-1} + (1-p)dV_{i+1,j+1})$$

and when $j = 0$

$$V_{i0} = e^{-r\Delta t}(puV_{i+1,0} + (1-p)dV_{i+1,1}).$$

We do this for $i = n-1, \ldots, 0$ finally obtaining V_{00}. Then the value of the option at time 0 is

$$S_0 V_{00}.$$

Convergence:

If we let $C(n)$ be the price of the lookback put with floating strike computed by the binomial model as above, it turns out that there is a constant A such that

$$C(n) = C_{BS} + \frac{A}{\sqrt{n}} + O\left(\frac{1}{n}\right)$$

so that convergence is rather slow.

However we can accelerate by extrapolation:

$$2C(2n) - C(n) = C_{BS} + O\left(\frac{1}{n}\right).$$

Hull-White binomial method for Asian options

Asian Option with Floating Strike

The payoff functions are given by, respectively for the Asian call and the Asian put:

$$AC_{\text{float}} = \max\left\{S_T - \frac{1}{T}\int_0^T S_u du, 0\right\},$$

$$AP_{\text{float}} = \max\left\{\frac{1}{T}\int_0^T S_u du - S_T, 0\right\}.$$

Asian Option With Fixed Strike

The payoff functions are given by, respectively for the Asian call and the Asian put:

$$AC_{\text{fixed}} = \max\left\{\frac{1}{T}\int_0^T S_u du - K, 0\right\},$$

$$AP_{\text{fixed}} = \max\left\{K - \frac{1}{T}\int_0^T S_u du, 0\right\}.$$

In the case of Asian options, there is no convenient closed form formula for the expectation $e^{-r(T-t)}E_Q(C_T|\mathcal{F}_t)$. Other methods can be used to approximate the value of the option. These include analytic approximations, lattice methods such as the binomial model, partial differential equation approaches and Monte Carlo methods. Here we focus on the binomial model.

We follow the article by Dai, Huang and Lyuu. This is an extension of an algorithm due to Hull and White. The idea here is that the value of the option depends on the current stock price S_t and the average A_t up to time t. However at a node in the stock price tree there will be many possible averages. Hull and White's idea is to calculate the option price just for a finite number of possible averages and use linear interpolation when necessary.

The three main ideas in the paper by Dai, Huang and Lyuu are:

(1) If the average exceeds a certain value, then the option always finishes in the money and we have an exact formula for its value.

(2) We choose the number of averages at each node to be proportional to the square root of the number of periods. The number of averages at each mode is also chosen to minimize the error caused by interpolation. It turns out that the combined error, due to the binomial approximation together with the interpolation, is of order $1/n$ and that the time of computation is order $n^{2.5}$.

(3) It turns out that the convergence of the price as calculated by this algorithm is smooth in the number of periods so that extrapolation can be used to accelerate the convergence. So good accuracy can be achieved with a relatively small n.

Pricing Asian Options in The Binomial Model

We divide the time interval $[0, T]$ into n intervals of length $\Delta t = T/n$. We write $t_i = i\Delta t$ and let $S(t_i)$ be the stock price at time t_i. We use the CRR (Cox-Rubinstein-Ross) lattice so that

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = u^{-1}.$$

The risk neutral probability is

$$p = \frac{e^{r\Delta t} - d}{u - d}.$$

Then, in the risk-neutral world, at time t_{i+1}, the stock price is $S(t_i)u$ with probability p and $S(t_i)d$ with probability $1 - p$.

Asian options are path-independent options. At time T the value is known and either just depends on

$$A(T) = \frac{1}{n+1}\sum_{i=0}^{n} S(t_i)$$

or on $A(T)$ and $S(T)$. At time t_i the value is given by a function $v(t_i, S(t_i), A(t_i))$, where

$$A(t_i) = \frac{1}{i+1}\sum_{j=0}^{i} S(t_j).$$

v can be calculated using the recurrence

$$v(t_i, S(t_i), A(t_i))$$

$$= e^{-r\Delta t}\{pv(t_{i+1}, S(t_i)u, [(i+1)A(t_i) + S(t_i)u]/(i+2))$$

$$+(1-p)v(t_{i+1}, S(t_i)d, [(i+1)A(t_i) + S(t_i)d]/(i+2)\}.$$

Unfortunately, the number of averages at any given node can be almost as many as the number of paths. In general the computing time required would be exponential.

The idea of Hull and White is to consider a finite number of averages $A(t_i)$ at each node. However then the problem is that the averages

$$A_u = [(i+1)A(t_i) + S(t_i)u]/(i+2),$$

$$A_d = [(i+1)A(t_i) + S(t_i)d]/(i+2)$$

at the next period may not be one of the averages for which we have computed the value of the option. The idea then is to use linear interpolation.

The Algorithm

At each node, where $S_{ij} = S_0 u^{i-j} d^j$, we only consider a finite number $k_{ij} + 1$ of averages restricted to a certain interval $[0, U_i]$. k_{ij} may vary from node to node.

U_i is chosen so that when $A(t_i) \geq U_i$, $v(t_i, S_{ij}, A(t_i))$ can be computed exactly. It turns out we can take

$$U_i = (n + 1)K/(i + 1).$$

This is because of the following lemma, due to Dai, Huang and Lyuu:

Lemma. *Suppose*

$$A(t_i) = \sum_{j=0}^{i} S(t_j) \geq (n+1)K.$$

Then

$$v(t_i, S_{ij}, A(t_i))$$

$$= \frac{e^{-r(n-i)\Delta t}}{n+1} \left[(i+1)A(t_i) - (n+1)K + S_{ij} e^{r\Delta t} \frac{1 - e^{(n-i)r\Delta t}}{1 - e^{r\Delta t}} \right].$$

The Algorithm: First we describe the algorithm, assuming the k_{ij} are given.

At the initial node, we just calculate $v(0, S_0, S_0)$ so that $k_{00} = 1$. Also at maturity, we can calculate the option exactly.

We describe how the other k_{ij}'s are defined later.

We compute $v(t_i, S(t_i), A)$ by backward recursion. It is convenient to write

$$v(i, j, \ell) = v(t_i, S_{ij}, A_{ij\ell}),$$

where $A_{ij\ell} = \ell U_i / k_{ij}$, $\ell = 0, \ldots, k_{ij}$ are the averages for which we are calculating the option value at the node where $S_{ij} = S_0 u^{i-j} d^j$.

Note: sometimes we write $v(i, j, A)$ to mean $v(t_i, S_{ij}, A)$.

From the payoff function, we know $v(n, j, A)$ for $j = 0, \ldots, n$ and any A. In fact, in the case of a call with fixed strike as we are considering here,

$$v(n, j, A) = \max(A - K, 0).$$

Suppose we have already calculated $v(i+1, j, \ell)$ for $\ell = 0, \ldots, k_{i+1,j}$, $j = 0, \ldots, i+1$. Then by risk-neutral valuation,

$$v(i, j, \ell)$$
$$= e^{-r\Delta t}[pv(i+1, j, A_u) + (1-p)v(i+1, j+1, A_d)$$

for $j = 0, \ldots, i$, where

$$A_u = \frac{(i+1)A_{ij\ell} + S_{ij}u}{i+2}, \quad A_d = \frac{(i+1)A_{ij\ell} + S_{ij}d}{i+2}.$$

When $i = n-1$,

$$v(i+1, j, A_u) = \max(A_u - K, 0),$$

$$v(i+1, j+1, A_d) = \max(A_d - K, 0).$$

When $i < n-1$, A_u and A_d may not be values of A for which we have calculated $v(i+1, j, A)$ or $v(i+1, j+1, A)$.

For A_u there are three possibilities: (1) $A_u > U_{i+1}$; (2) $A_u < U_{i+1}$; (3) $A_u = U_{i+1}$.

In Case (1), we use Lemma 1 to calculate $v(i+1, j, A_u)$.

In Case (2) we use linear interpolation. We find m with $0 \le m < k_{i+1,j}$ such that

$$m\frac{U_{i+1}}{k_{i+1,j}} \le A_u < (m+1)\frac{U_{i+1}}{k_{i+1,j}},$$

that is

$$m \le \frac{k_{i+1,j}A_u}{U_{i+1}} < m+1.$$

So

$$m = \text{floor}\left(\frac{k_{i+1,j}A_u}{U_{i+1}}\right).$$

If we define

$$a = \frac{k_{i+1,j}A_u}{U_{i+1}} - m,$$

we see that

$$A_u = (1-a)\frac{mU_{i+1}}{k_{i+1,j}} + a\frac{(m+1)U_{i+1}}{k_{i+1,j}}$$

so that the linear interpolated value is

$$v(i+1,j,A_u) = (1-a)v(i+1,j,m) + av(i+1,j,m+1).$$

In Case (3)

$$v(i+1, j, A_u) = v(i+1, j, k_{i+1,j}).$$

For A_d there are also three possibilities: (1)$A_d > U_{i+1}$; (2) $A_d < U_{i+1}$; (3) $A_d = U_{i+1}$.

These are handled similarly.

We do this until $i = 1$. When $i = 0$, we only need to calculate $v(0, 0, A)$ with $A = S_0$. The final option value is just this value.

Choosing k_{ij} to minimize the interpolation error

Determination of k_{ij}

First we find an upper bound for the interpolation error. Then we choose the k_{ij} to minimize this upper bound.

Step 1: Upper bound for the error

We want to find an upper bound for the error occurring when we approximate $v(i+1, j, A_u)$ by

$$(1-a)v(i+1, j, m) + av(i+1, j, m+1),$$

where m corresponds to $A''_u = \frac{mU_{i+1}}{k_{i+1,j}}$ and $m+1$ to $A'_u = \frac{(m+1)U_{i+1}}{k_{i+1,j}}$, and similarly for A_d.

Recall that

$$A''_u = m\frac{U_{i+1}}{k_{i+1,j}} \le A_u < A'_u = (m+1)\frac{U_{i+1}}{k_{i+1,j}},$$

so that

$$|A_u - A''_u|,\ |A_u - A'_u| \le \frac{U_{i+1}}{k_{i+1,j}}.$$

The exact option value is a function $v(t, S, A_u)$. Here $t = t_{i+1}$, $S = S_0 u^{i+1-j} d^j$ and we approximate $v(t, S, A_u)$ by

$$(1-a)v(t, S, A''_u) + av(t, S, A'_u),$$

where a is such that

$$A_u = (1-a)A''_u + aA'_u.$$

Assuming $v(t, S, A)$ is a smooth function, the difference

$$v(t, S, A_u) - [(1-a)v(t, S, A''_u) + av(t, S, A'_u)]$$

$$= \frac{(A_u - A''_u)(A'_u - A_u)}{2}\frac{\partial^2 v}{\partial A^2}(t, S, \eta),$$

where η is between A''_u and A'_u.

Then we conclude

$$|v(t,S,A_u) - [(1-a)v(t,S,A_u'') + av(t,S,A_u')]|$$

$$\leq M \left[\frac{U_{i+1}}{k_{i+1,j}}\right]^2.$$

Similarly for A_d, when $t = t_{i+1}$, $S = S_0 u^{i-j} d^{j+1}$, we approximate $v(t,S,A_d)$ by

$$(1-a)v(t,S,A_d'') + av(t,S,A_d'),$$

where

$$A_d'' = m\frac{U_{i+1}}{k_{i+1,j+1}} \leq A_d < A_d' = (m+1)\frac{U_{i+1}}{k_{i+1,j+1}}$$

and a is such that

$$A_d = (1-a)A_d'' + aA_d'.$$

So

$$|v(t,S,A_d) - [(1-a)v(t,S,A_d'') + av(t,S,A_d')]|$$

$$\leq M \left[\frac{U_{i+1}}{k_{i+1,j+1}}\right]^2.$$

Let $\overline{v}(i, j, A)$ be the computed value at the node where $S_{ij} = S_0 u^{i-j} d^j$ for $A = A_{ij}$.

After a recursion, we find that

$$\begin{aligned} & |\overline{v}(0, 0, S_0) - v(0, 0, S_0)| \\ & \leq K^2 M \sum_{i=1}^{n-1} \sum_{j=0}^{i} \frac{B(i, j; p)(n+1)^2}{i^2 k_{ij}^2}, \end{aligned} \quad (1)$$

where

$$B(i, j; p) = \binom{i}{j} p^{i-j} (1-p)^j.$$

Step 2: *Using Lagrange multipliers to determine k_{ij}*

For a given k, we choose k_{ij} to minimize the function

$$f(k_{ij}) = \sum_{i=1}^{n-1} \sum_{j=0}^{i} \frac{B(i,j;p)}{i^2 k_{ij}^2}$$

subject to

$$g(k_{ij}) = \sum_{i=1}^{n-1} \sum_{j=0}^{i} k_{ij} = (n-1)^2 k/2.$$

This means the average k_{ij} is about k. We use the method of Lagrange multipliers and find that

$$k_{ij} = \frac{(n-1)^2 k}{2} \frac{\left[\frac{B(i,j;p)}{i^2}\right]^{1/3}}{\sum_{\ell=1}^{n-1} \sum_{m=0}^{\ell} \left[\frac{B(\ell,m;p)}{\ell^2}\right]^{1/3}} \quad (2)$$

for $1 \leq i \leq n-1$, $0 \leq j \leq i$.

Estimation of interpolation error

Next we show that if we choose k_{ij} as in the previous subsection, the interpolation error is of order $1/k^2$, where k is the average of the k_{ij}'s.

From (1), our minimized error $|\bar{v}(0,0,S_0) - v(0,0,S_0)|$ is bounded by

$$M(n+1)^2K^2 \sum_{i=1}^{n-1} \sum_{j=0}^{i} \frac{B(i,j;p)}{i^2 k_{ij}^2}$$

with k_{ij} as in (2). Let us write

$$b_{ij} = \left[\frac{B(i,j;p)}{i^2}\right]^{1/3}, \quad S = \sum_{\text{I}=1}^{n-1} \sum_{j=0}^{i} b_{ij}$$

so that

$$k_{ij} = \frac{(n-1)^2 k}{2} b_{ij}/S.$$

Then our error is bounded by

$$S^2 \frac{4M(n+1)^2K^2}{(n-1)^4k^2} \sum_{i=1}^{n-1} \sum_{j=0}^{i} \frac{B(i,j;p)}{i^2 b_{ij}^2}.$$

which equals

$$S^2 \frac{4M(n+1)^2K^2}{(n-1)^4k^2} \sum_{i=1}^{n-1} \sum_{j=0}^{i} b_{ij} \leq S^3 \frac{4M(n+1)^2K^2}{(n-1)^4k^2}.$$

Then we determine the asymptotic behaviour of S:

$$S \sim n^{2/3}.$$

So the error $|\bar{v}(0,0,S_0) - v(0,0,S_0)|$ is bounded by

$$S^3 \frac{4M(n+1)^2K^2}{(n-1)^4k^2},$$

which is of order $1/k^2$. This is a bound for the interpolation error.

Estimation of total error and computation time

We also have the error from approximating the Black-Scholes price by the binomial price which Forsythe et al argue is of order $1/n$.

So if we take k, the average of the k_{ij}'s, to be of order $\sqrt{n}$, then it follows from the conclusion of the previous paragraph that the total error is of order $1/n$, which we observe in examples.

Then the convergence time is of order $n^2k = n^{2.5}$.

Smooth Convergence

Let $C(n)$ be the binomial price when the number of periods is n. Then it should be true that

$$C(n) \to C_{BS} \quad \text{as} \quad n \to \infty.$$

If we look at the numerical results we find there is a constant D such that

$$C(n) = C_{BS} + \frac{D}{n} + o\left(\frac{1}{n}\right). \tag{3}$$

Note when (3) holds, that

$$n(C(n) - C_{BS}) \to D \quad \text{as} \quad n \to \infty.$$

So $C(n)$ is a linear function of $1/n$ and D is the slope of the line. Then we can use Richardson extrapolation to accelerate the convergence. We have

$$C(n) = C_{BS} + \frac{D}{n} + o\left(\frac{1}{n}\right), \quad C(2n) = C_{BS} + \frac{D}{2n} + o\left(\frac{1}{n}\right)$$

So

$$2C(2n) - C(n) = C_{BS} + o\left(\frac{1}{n}\right).$$