Review of basic concepts: arbitrage, options

In this lecture we review the basic notion of arbitrage. We also define call and put options and derive arbitrage bounds for their prices; we also study put-call parity.

Arbitrage

If I can purchase a portfolio for zero cost at time 0 and then at time T, I never lose but there is a positive probability that I gain

or

if I can purchase a portfolio for negative cost at time 0 and then at time T I never lose

then this is said to be an *arbitrage opportunity*.

In an *efficient market*, arbitrage opportunities should not exist. We assume markets are efficient.

Risk-Free Rate

If we can deposit \$1 in the bank at time 0 and at time T we obtain \$$e^{rT}$, then r is said to be the (continuously compounded) risk-free rate. We assume that we can both lend and borrow money at this rate. Arbitrage arguments imply that there can be only one risk-free rate.

European Call and Put Options

Definition A *European call option* gives the holder the *right*, but not the *obligation*, to *buy* an asset at a specified time T for a specified price K.

A *European put option* gives the holder the *right*, but not the *obligation*, to *sell* an asset at a specified time T for a specified price K.

T: exercise date or maturity or expiry date

K: strike price or exercise price

Payoffs to Options

The payoff to a call option is $\max\{S_T - K, 0\}$.

The payoff to a put option is $\max\{K - S_T, 0\}$.

Put-Call Parity

If the current stock price is S, the price of a European call with strike K and maturity T is C and the price of a European put with strike K and maturity T is P, then

$$S + P - C = Ke^{-rT},$$

where r is the risk-free rate.

Proof. Let Π be the portfolio consisting of one share, one put and one written call. At $t = 0$, this costs $S + P - C$. At time T, the payoff to Π is $S_T + 0 - (S_T - K) = K$ if $S_T \geq K$ and $S_T + (K - S_T) - 0 = K$ if $S_T < K$. Thus the payoff to Π is the same as the payoff to Ke^{-rT} of cash bond. Hence, in the absence of arbitrage, we must have $S + P - C = Ke^{-rT}$.

Arbitrage Bounds

If the current stock price is S, then the price C of a European call with strike K and maturity T satisfies

$$C \le S, \quad C \ge \max\{S - Ke^{-rT}, 0\},$$

where r is the risk-free rate.

If the current stock price is S, then the price P of a European put with strike K and maturity T satisfies

$$P \le Ke^{-rT}, \quad P \ge \max\{Ke^{-rT} - S, 0\},$$

where r is the risk-free rate.

Different Strikes

Consider European call options on the same underlying and with the same maturity but different strike prices. Then if $K_2 \geq K_1$ *and* $0 \leq \lambda \leq 1$,

$$e^{-rT}(K_2 - K_1) \geq C(K_1) - C(K_2) \geq 0;$$

$$\lambda C(K_1) + (1-\lambda)C(K_2) \geq C(\lambda K_1 + (1-\lambda)K_2).$$

(Question: what are the corresponding relations for puts?)

Different Maturities

Consider European call options on the same underlying and with the same strike price K *but different maturities. Then if* $T_2 \geq T_1$,

$$C(T_2) \geq C(T_1).$$

(Note: this does not hold for European puts.)

American Options

Definition. An *American call option* with strike price K and expiry time T gives the holder the right, but not the obligation, to buy an asset for price K *at any time up to* T.

An *American put option* with strike price K and expiry time T gives the holder the right, but not the obligation, to sell an asset for price K *at any time up to* T.

Lemma. *It is never optimal to exercise an American call option on a non-dividend paying stock before expiry.*

Risk neutral valuation of options in one-period discrete-time markets

In this lecture we study one-period markets and show the fundamental theorem that such markets are arbitrage-free if and only there exists a martingale measure (or risk-neutral probability).

We also show that the price of an attainable contingent claim is the discounted expectation of its payoff under a martingale measure.

The notion of completeness is also introduced.

General One-Period Markets

Suppose there are N securities in the market. Their prices at $t = 0$ are given by

$$S_0 = (S_0^1, S_0^2, \ldots, S_0^N)^t = \begin{bmatrix} S_0^1 \\ S_0^2 \\ \cdot \\ \cdot \\ \cdot \\ S_0^N \end{bmatrix}.$$

At time T there are n possible states. The value of the ith security at time T in state j is given by D_{ij}.

The $N \times n$ matrix $D = [D_{ij}]$ is called the *payoff matrix*.

Definition. A *portfolio* is a vector $\theta = (\theta_1, \theta_2, \ldots, \theta_N)^t$ where θ_i is the number of units of the ith security. So the market value at $t = 0$ of the portfolio is

$$\theta_1 S_0^1 + \theta_2 S_0^2 + \cdots + \theta_N S_0^N = S_0 \cdot \theta = S_0^t \theta.$$

The value of the portfolio at time T in state j is

$$D_{1j}\theta_1 + D_{2j}\theta_2 + \cdots + D_{Nj}\theta_N,$$

which is the jth component of the vector $D^t\theta$.

Definition. An *arbitrage opportunity* is a portfolio θ with either

$$S_0 \cdot \theta = 0, \quad D^t\theta > 0, \quad \text{or} \quad S_0 \cdot \theta < 0, \quad D^t\theta \geq 0.$$

(Note: if x is a vector $x \geq 0$ means all components of x are nonnegative; $x > 0$ means all components of x are nonnegative and at least one is positive; $x >> 0$ means all components of x are positive.)

Definition. A *state price vector* is a vector $\psi >> 0$ in $\mathcal{R}^n$ such that $S_0 = D\psi$.

(Suppose I can find a portfolio θ which delivers the payoff \$1 in state j and zero in the other states. So $D^t\theta = e_j$, where e_j is the vector with jth component 1 and all other components zero. If there is a state price vector ψ, this portfolio would cost $S_0^t\theta = \psi^t D^t\theta = \psi^t e_j = \psi_j$. That is, ψ_j is the price of the portfolio which delivers the payoff \$1 in state j and zero in the other states.)

Theorem. *There is no arbitrage opportunity if and only if there is a state price vector.*

Proof. We just prove the "if" part. Suppose there is a vector $\psi >> 0$ in $\mathcal{R}^n$ such that $S_0 = D\psi$. Now suppose there is an arbitrage opportunity, that is, a portfolio θ with either

$$S_0 \cdot \theta = 0, \quad D^t\theta > 0, \quad \text{or} \quad S_0 \cdot \theta < 0, \quad D^t\theta \geq 0.$$

Now $S_0 \cdot \theta = \psi^t D^t \theta$. This means if $D^t\theta > 0$, then $S_0 \cdot \theta > 0$ and if $D^t\theta \geq 0$, then $S_0 \cdot \theta \geq 0$. So there is no arbitrage opportunity.

Example 1 (The Binomial Model): Here we have a stock and a risk-free asset, where we assume that the stock price is S now and goes up to Su or down to Sd at time T ($u > d$), and \$1 invested in the risk-free asset yields $R = e^{rT}$. So we have the table

State	1	2
S_1	Su	Sd
S_2	R	R

The state prices ψ_1, ψ_2 satisfy

$$Su\psi_1 + Sd\psi_2 = S, \quad R\psi_1 + R\psi_2 = 1.$$

and so

$$\psi_1 = \frac{R-d}{R(u-d)}, \quad \psi_2 = \frac{u-R}{R(u-d)}.$$

Hence there is no arbitrage if and only if $u > R > d$.

Pricing Contingent Claims in a One-Period Model

Definition. A *contingent claim* is a vector of payoffs $C = (C_1, C_2, \dots, C_n)^t$. C is said to be *attainable* if there is a portfolio θ with payoff $D^t\theta = C$. The portfolio θ is said to *replicate* the claim C.

Suppose our market has no arbitrage opportunities. Then there is a state price vector ψ. Then the cost of the portfolio with payoff C is

$$S_0^t\theta = \psi^t D^t\theta = \psi^t C.$$

That is, *the cost of an attainable contingent claim C in a market with no arbitrage opportunities is $\psi^t C$, where ψ is a state price vector.*

Facts. (i) If C is an attainable claim in an arbitrage-free market, then $S_0^t\theta$ is independent of θ such that $D^t\theta = C$;

(ii) if C is an attainable claim in an arbitrage-free market, then $\psi^t C$ does not depend on ψ.

Theorem. *A contingent claim C in a one-period arbitrage-free market is attainable if and only if $\psi^t C$ is the same value for all state price vectors ψ.*

Proof. We already know that if a contingent claim C is attainable, then $\psi^t C$ is the same value for all state price vectors ψ.

Suppose, conversely, that $\psi^t C$ is the same value for all state price vectors ψ. Now C is attainable if and only if there exists θ such that

$$D^t \theta = C.$$

By a well-known theorem from linear algebra, such a θ exists if and only if whenever b is a vector with $b^t D^t = 0$, then also $b^t C = 0$. So suppose $b^t D^t = 0$. Then $Db = 0$. Let ψ be a state price vector so that $\psi >> 0$ and $D\psi = S_0$. For sufficiently small positive ε, $\psi + \varepsilon b >> 0$ and $D(\psi + \varepsilon b) = S_0$ so that $\psi + \varepsilon b$ is also a state price vector. By hypothesis, $\psi^t C = (\psi + \varepsilon b)^t C$. Hence $b^t C = 0$ and we conclude that C is attainable.

Completeness

Definition. A market is said to be *complete* if every contingent claim is attainable.

In general, *a securities market is complete if and only if the payoff matrix has rank* n, *where* n *is the number of states*.

Theorem. *Suppose our market is arbitrage-free so that there exists a state price vector. Then the market is complete if and only if this state price vector is unique.*

Proof. Suppose first the market is complete. Let

$$\psi = (\psi_1, \psi_2, \ldots, \psi_n)^t \quad \text{and} \quad \tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_n)^t$$

be two state price vectors. Any contingent claim C is attainable. So there is a portfolio θ such that $D^t\theta = C$. The price of this portfolio is $S_0^t\theta$, which must be independent of θ (otherwise there would be an arbitrage opportunity). However we also know this price is given by $\psi^t C$ and $\tilde{\psi}^t C$. Hence

$$\psi^t C = \tilde{\psi}^t C$$

for all C. This implies $\psi = \tilde{\psi}$.

Suppose, conversely, that the state price vector ψ is unique. But then if C is a contingent claim, $\psi^t C$ has only one possible value. Hence by the previous theorem, C is attainable. So the market is complete.

We summarize:

The market is arbitrage-free if and only if there exists a state price vector ψ*.*

In addition, the market is complete if and only if this state price vector ψ *is unique.*

In an arbitrage-free market the price of an attainable claim C *is* $\psi^t C$*.*

Example 2: The Binomial Model: Here we have a stock and a risk-free asset, where we assume that the stock price is S now and goes up to Su or down to Sd at time T ($u > d$), and \$1 invested in the risk-free asset yields $R = e^{rT}$. So we have the table

State	1	2
S_1	Su	Sd
S_2	R	R

As found above, the market is arbitrage-free if and only if $d < R < u$. The market is complete since the payoff matrix

$$\begin{bmatrix} Su & Sd \\ R & R \end{bmatrix}$$

has rank 2. This is consistent with the fact found above that there is a unique state price vector $\left(\frac{R-d}{R(u-d)}, \frac{u-R}{R(u-d)}\right)$.

Risk-Neutral Probability Measures

Definition. A portfolio is *risk-free* if it has the same payoff \$1 in all states.

The risk-free portfolio can be thought as a bank account. Let its price be ψ_0. If at $t = 0$ I pay ψ_0, at time T I obtain \$1. ψ_0 is the *discount factor*; \$1 at time T is worth ψ_0 at time $t = 0$.

Definition. If the market has n possible states at time T and has a risk-free portfolio (with price ψ_0), then any vector $p = (p_1, p_2, \ldots, p_n)$ of probabilities (i.e, $p_j > 0$ for all j and $p_1 + p_2 + \cdots + p_n = 1$) for which each security's price is its discounted expected payoff (that is, $S_0^i = \psi_0 \sum_{j=1}^{n} p_j D_{ij}$ for $i = 1, ..., N$) is called a *risk-neutral probability measure* or *equivalent martingale measure*.

Claim. *If the market has a risk-free portfolio with price* ψ_0, *and* $p = (p_1, p_2, \ldots, p_n)$ *is a risk-neutral probability measure, then the price of the portfolio with payoff* $C = (C_1, ..., C_n)$ *is*

$$\psi_0 \sum_{j=1}^{n} p_j C_j = \psi_0 E(C).$$

Proof. If the portfolio is θ, its payoff in state j is

$$C_j = \sum_{i=1}^{N} \theta_i D_{ij}$$

and its price is

$$\sum_{i=1}^{N} \theta_i S_0^i = \psi_0 \sum_{i=1}^{N} \theta_i \sum_{j=1}^{N} p_j D_{ij} = \psi_0 \sum_{j=1}^{n} p_j \sum_{i=1}^{N} \theta_i D_{ij}$$

which equals

$$\psi_0 \sum_{j=1}^{n} p_j C_j.$$

Claim. *Suppose the market has a risk-free portfolio with price ψ_0. Then if ψ is a state price vector,*

$$\psi_0 = \psi_1 + \cdots + \psi_n$$

and the numbers

$$p_j = \psi_j/\psi_0, \quad j = 1, .., n$$

form a risk-neutral probability measure. Conversely, if $p = (p_1, ..., p_n)$ is a risk-neutral probability measure, then the numbers

$$\psi_j = \psi_0 p_j, \quad j = 1, .., n$$

form a state price vector.

Proof. Let ψ be a state price vector. By a Theorem above, the price of the risk-free portfolio which has payoff $C = (1, \ldots, 1)$ is

$$\psi^t C = \psi_1 + \cdots + \psi_n.$$

So $\psi_0 = \psi_1 + \cdots + \psi_n$. Next we know that $D\psi = S_0$ so that

$$S_0^i = \sum_{j=1}^{n} D_{ij}\psi_j$$

for all i. Thus

$$S_0^i = \psi_0 \sum_{j=1}^{n} p_j D_{ij}$$

for all i, where for $j = 1, ..., n$

$$p_j = \frac{\psi_j}{\psi_0}.$$

Conversely, suppose $p = (p_1, ..., p_n)$ is a risk-neutral probability measure. Then

$$S_0^i = \psi_0 \sum_{j=1}^{n} p_j D_{ij}$$

for all i. It follows that

$$S_0^i = \sum_{j=1}^{n} D_{ij} \psi_j$$

for all i, where for $j = 1, ..., n$

$$\psi_j = \psi_0 p_j.$$

That is, $D\psi = S_0$.

Note that the risk-neutral probabilities p_j may have nothing to do with the "real world" distribution.

We summarize:

Suppose the market has a risk-free portfolio. Then

it is arbitrage-free if and only if there exists a risk-neutral probability measure p*;*

if the market is arbitrage-free, then it is complete if and only if this risk-neutral probability measure p *is unique;*

In an arbitrage-free market, the price of an attainable claim C *is* $\psi_0 p^t C = \psi_0 E[C]$*.*

Note: if the market is complete, it has a risk-free portfolio. If the market is arbitrage-free but incomplete, it may or may not have a risk-free security.

Example 3: The Binomial Model: Here we have a stock and a risk-free asset, where we assume that the stock price is S now and goes up to Su or down to Sd at time T ($u > d$), and \$1 invested in the risk-free asset yields $R = e^{rT}$. So we have the table

State	1	2
S_1	Su	Sd
S_2	R	R

The state prices are

$$\psi_1 = \frac{R-d}{R(u-d)}, \quad \psi_2 = \frac{u-R}{R(u-d)}.$$

and there is no arbitrage if and only if $u > R > d$. The risk-neutral probabilities are $p_1 = (R-d)/(u-d)$ and $p_2 = (u-R)/(u-d)$ and the attainable contingent claim which pays C_1 in state 1 and C_2 in state 2 has price

$$\frac{p_1 C_1 + p_2 C_2}{R}.$$