# Low regularity solutions for the one dimensional Schrödinger equations with derivative

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Start

# The model:

The Cauchy problem for the one dimensional Schrödinger equations with derivative

$$\begin{cases} v_t - iv_{xx} = (|v|^2 v)_x, (x,t) \in \mathbb{R} \times \mathbb{R} \\ v(x,0) = v_0(x) \in H^s(\mathbb{R}). \end{cases} \tag{1}$$

where the unknown function $v$ is complex valued.

# Background:

The equation in (1) is a model of the propagation of circularly polarized Alfvèn waves in magnetized plasma with a constant external magnetic field.

# Known results:

- Well-posedness:

1. Guo-Tan (1991) showed smooth solution for (1), Proc. Roy. Soc. Edinburgh, 119A, 31-45.

2. N. Hayashi-T. Ozawa (1992) showed global well-posedness in $H^1$ solution for (1) with small data in $L^2$, Physica D: Nonlinear Phenomena, 55, 14-36.

3. Guo-WU (1995), Orbital stability of solitary waves for NDSE, J. Diff. Equ., 123, 35-55.

4. H. Takaoka (1999) showed local well-posedness in $H^{1/2}$ for (1), Adv. Diff. Equ., 4, 561-680.

5. Colliander, Keel, Staffilani, Takaoka and Tao (2002) showed that under smallness condition on initial data in $L^2$, the Cauchy problem (1) is globally well-posed in $H^s$, $s > \frac{1}{2}$, SIAM J. Math. Anal., 34(1), 64-86.

6. C. Miao, Y. Wu and G. Xu (2011) showed that under smallness condition on initial data in $L^2$, the Cauchy problem (1) is globally well-posed in $H^{1/2}$, J. Diff. Equ., 251, 2164-2195.

- Ill-posedness

Define data-map $A(t)$ of (1) by $v(t) = A(t)v_0$.

H. Takaoka (2001) showed that if data-map $A(t)$ of (1) is $C^3$, the Cauchy problem is ill-posed in $H^s$ for $s < 1/2$.

# Problem:

The scaling argument suggests the value of $L^2$ critical for the local well-posedness. Thereby we see the gap between the suggestions of scaling argument and of the local well-posedness results in $H^{1/2}$.

We will answer this gap in this lecture.

# Our results:

**Theorem 1(G-Huo,2013).** *The Cauchy problem (1) is locally well-posed in $H^s$ with $s \geq \frac{1}{3}$ assuming the smallness condition on initial data in the $L^2$ norm.*

**Theorem 2(G-Huo,2013).** *The Cauchy problem (1) is globally well-posed in $H^s$ with $s \geq \frac{-3+\sqrt{41}}{8} \approx 0.425$ assuming the smallness condition on initial data in the $L^2$ norm.*

**Remark.** Compared with Kato's conjecture(1983), the index $s = 1/3$ is sharp. For global result, there exists a gap between $1/3$ and $\frac{-3+\sqrt{41}}{8}$. Recently, we find that it is possible to obtain global well-posedness in $H^{1/3}$.

Kato's conjecture(1983): Kato conjectured modified Korteweg-de Vries equation

$$\partial_t w + \partial_x^3 w + \partial_x (w^3) = 0, w(0) = w_0$$

is globally well-posed in $L^2$. The index $L^2$ is sharp. In fact, we completed this conjecture. The index $L^2$ in mKdV equation correspond with the index $H^{1/3}$ in equation (1) using the method of modulated wave train solutions.

# The Key difficulity for $H^s$ with $s < 1/2$:

Usually, via the Gauge transformation

$$u = v(x,t)e^{-i\int_{-\infty}^{x}|v(y,t)|^2dy}$$

(1) is formally rewritten as the Cauchy problem

$$\begin{cases} u_t - iu_{xx} = -u^2\bar{u}_x + \frac{i}{2}|u|^4u, \\ u(x,0) = u_0(x) \in H^s(\mathbb{R}). \end{cases} \tag{2}$$

where

$$u_0 = v_0(x)e^{-i\int_{-\infty}^{x}|v_0(y)|^2dy}.$$

The Cauchy problem (2) is interesting because of the derivative in the nonlinearity has been removed: $|u|^2u_x$ in (1) has been replaced by the quintic nonlinearity $|u|^4u$ in (2). The Strichartz estimate can control the nonlinearity $|u|^4u$ easy.

The term $u^2\bar{u}_x$ can be controlled by the standard Bourgain spaces $X_{s,b}$

$$\|u\|_{X_{s,b}} = \|\langle\xi\rangle^s\langle\tau-\xi^2\rangle^b\widehat{u}(\xi,\tau)\|_{L^2_\xi L^2_\tau}; \tag{3}$$

- The method of standard Bourgain spaces:

**Lemma 1 (Linear estimates,[Takaoka (1999)]).** *Let* $s \in \mathbb{R}, \frac{1}{2} < b < 1$. *The group of Schrödinger equation* $S(t) = e^{i\triangle t}$ *in one spatial dimension satisfies*

$$\|S(t)\varphi\|_{X_{s,b}} \leq C\|\varphi\|_{H^s}. \tag{4}$$

$$\left\|\int_0^t S(t-t')f(t')dt'\right\|_{X_{s,b}} \leq C\|f\|_{X_{s,b-1}}. \tag{5}$$

**Lemma 2 (Non-linear estimates, [Takaoka (1999)]).** *Let* $s \geq 1/2, \frac{1}{2} < b < 1$.

$$\|u^2\bar{u}_x\|_{X_{s,b-1}} \lesssim \|u\|^3_{X_{s,b}}. \tag{6}$$

It is proved that the contraction argument provides the local well-posedness of (2) in $H^s$ with $s \geq 1/2$ using Lemmas 1 and 2.

**Outline of the proof of Lemma 2:**

By duality and the Plancheral identity, it suffices to show

$$\int_{\Gamma_4} \frac{\langle \xi_4 \rangle^s \langle \tau_4 - \xi_4^2 \rangle^{b-1} \xi_3 \prod_{i=1}^4 f_i(\xi_i, \tau_i)}{\langle \tau_1 + \xi_1^2 \rangle^b \langle \tau_2 + \xi_2^2 \rangle^b \langle \tau_3 - \xi_3^2 \rangle^b \prod_{i=1}^3 \langle \xi_i \rangle^s} \lesssim \prod_{i=1}^4 \|f_i\|_{L^2}, \qquad (7)$$

for $f_i \in L^2(\mathbb{R}^2) \geq 0$, $i = 1, 2, 3, 4$.

$\Gamma_4 = \{(\xi, \tau) \in \mathbb{R}^4 \times \mathbb{R}^4 : \xi_1 + \xi_2 + \xi_3 + \xi_4 = 0, \tau_1 + \tau_2 + \tau_3 + \tau_4 = 0\}$

The bad case is :

$$|\xi_4| \sim |\xi_3| \sim |\xi_2| \sim |\xi_1|, \qquad (8)$$

which needs the condition $H^{1/2}$.

H. Takaoka (1999) also showed that for any $s < 1/2$ and any $b \in \mathbb{R}$, the estimate

$$\|u^2 \bar{u}_x\|_{X_{s,b-1}} \lesssim \|u\|^3_{X_{s,b}} \tag{9}$$

fails.

Therefore, it is difficult to obtain local well-posedness in $H^s$ with $s < 1/2$ by using directly contraction principle.

# Our method:

We will approach the problem in a less perturbative way. That is, we will use the known global result in $H^1$ to approach local result in $H^s$ with $s \geq 1/3$.

We first define suitable spaces $\mathbf{F}^s(T)$, $\mathbf{N}^s(T)$ and $\mathbf{E}^s(T)$, and show that if $u$ is a solution of the Cauchy problem (2) in $H^1$ for any $T > 0$, then for $s \geq 1/3$

$$\begin{cases} \|u\|_{\mathbf{F}^s(T)} \lesssim \|u\|_{\mathbf{E}^s(T)} + \|u^2\bar{u}_x\|_{\mathbf{N}^s(T)} + \||u|^4u\|_{\mathbf{N}^s(T)}; \\ \|u^2\bar{u}_x\|_{\mathbf{N}^s(T)} \lesssim \|u\|^3_{\mathbf{F}^s(T)}, \quad \||u|^4u\|_{\mathbf{N}^s(T)} \lesssim \|u\|^5_{\mathbf{F}^s(T)}; \\ \|u\|^2_{\mathbf{E}^s(T)} \lesssim \|u_0\|^2_{H^s} + \|u\|^4_{\mathbf{F}^s(T)} + \|u\|^6_{\mathbf{F}^s(T)} \end{cases} \tag{10}$$

From this, we can obtain the existence of the solution in $H^s$ with $s \geq 1/3$. To prove the uniqueness and continuity of solution, we need to consider the difference equation (2).

- Definitions, notations

First, we define the dyadic decomposition. Let $\eta : \mathbb{R} \to [0,1]$ denote an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For $j \in \mathbb{Z}$, let $\chi_j(\xi) = \eta(\xi/2^j) - \eta(\xi/2^{j-1})$, and

$$\chi_{[j_1,j_2]} = \sum_{j=j_1}^{j_2} \chi_j, \text{ and } \chi_{\leq j_2} = \sum_{j=0}^{j_2} \chi_j$$

For simplicity of notation, let $\eta_j = \chi_j$ if $j \geq 1$ and $\eta_0 = 1 - \sum_{j=0}^{+\infty} \eta_j$. Also, for $l_1 \leq l_2 \in \mathbb{Z}_+$

$$\eta_{[l_1,l_2]} = \sum_{l=l_1}^{l_2} \eta_l, \text{ and } \eta_{\leq l_2} = \sum_{l=0}^{l_2} \eta_1$$

For any integer $k \geq 0$, we define the operator $P_k$ with respective to the variable $x$ by the formula

$$\widehat{P_k u}(\xi) = \eta_k(\xi)\widehat{u}(\xi), \ \widehat{P_{\leq k} u}(\xi) = \eta_{\leq k}(\xi)\widehat{u}(\xi)$$

For $l \in \mathbb{Z}$, let $I_l = \{\xi \in \mathbb{R} : |\xi| \in [2^{l-1}, 2^{l+1}]\}$. For $l \in \mathbb{Z}_+$, let $\widetilde{I_l} = I_l$ if $k \geq 1$ and $\widetilde{I_0} = [-2, 2]$. For $k \in \mathbb{Z}_+$ and $j \geq 0$, let

$$D_{k,j} = \{(\xi, \tau) \in \mathbb{R} \times \mathbb{R} : \xi \in \widetilde{I_k}, \tau - \phi(\xi) \in \widetilde{I_j}\}$$

We define first the Banach spaces $X_k = X_k(\mathbb{R} \times \mathbb{R})$ for $k \in \mathbb{Z}_+$

$$X_k(\mathbb{R} \times \mathbb{R}) = \{f(\xi, \tau) \in L^2(\mathbb{R} \times \mathbb{R}) : f \text{ is supported in } \widetilde{I_k} \times \mathbb{R} \text{ and}$$
$$\|f\|_{X_k} = \sum_{j=0}^{+\infty} 2^{j/2} \alpha_{j,k} \|\eta_j(\tau - \phi(\xi)) f(\xi, \tau)\|_{L^2_{\xi,\tau}} < \infty, \} \tag{11}$$

$$\alpha_{j,k} := \begin{cases} 1 + 2^{(j-2k)/2}, & \text{if } j \gtrsim 3k, \\ 1 + 2^{(2k-j)/2} & \text{if } j \ll 2k. \end{cases} \tag{12}$$

For $k \in \mathbb{Z}_+$ define the frequency localized initial data spaces

$$E_k = \{\phi : \mathbb{R} \to \mathbb{R} : \mathcal{F}(\phi) = \eta_k(\xi)\mathcal{F}(\phi) \text{ and } \|\phi\|_{E_k} = \|\widehat{\phi}\|_{L^2_\xi} < \infty\}, \tag{13}$$

At frequency $2^{2k/3}$ we will use the $X^{s,b}$ structure given by the $X_k$ norm, uniformly on the $2^{-2k/3}$ time scale. For $k \in \mathbb{Z}$ we define the normed spaces

$$F_k = \{u_k \in C_0(\mathbb{R} : E_k) : \|u_k\|_{F_k} = \sup_{t_k \in \mathbb{R}} \|\mathcal{F}[u_k \cdot \eta_0(2^{2k/3}(t - t_k))]\|_{X_k} < \infty\}. \tag{14}$$

**Merit of $F_k$:** In the following, we can always assume that $j \geq \frac{2}{3}k$, which can deal with the index $s < 1/2$.

For $k \in \mathbb{Z}$ we define the normed spaces $N_k = C_0(\mathbb{R} : E_k)$, which are used to measure the frequency $2^{2k/3}$ part of the nonlinear term, with norms

$$\|f_k\|_{N_k} = \sup_{t_k \in \mathbb{R}} \|(\tau - \phi(\xi) + i2^{2k/3})^{-1} \cdot \mathcal{F}[f_k \cdot \eta_0(2^{k/3}(t - t_k))]\|_{X_k}, \tag{15}$$

In the following proof, we will use the localized spaces defined above, for any time $T \in (0,1]$ we define the normed spaces

$$\begin{cases} F_k(T) & = \{u_k \in C([-T,T]:E_k) : \|u_k\|_{F_k(T)} \\ & \qquad = \inf_{\widetilde{u}_k = u_k \text{ in } \mathbb{R}\times[-T,T]} \|\widetilde{u}_k\|_{F_k} < \infty\}; \\ N_k(T) & = \{f_k \in C([-T,T]:E_k) : \|f_k\|_{N_k(T)} \\ & \qquad = \inf_{\widetilde{f}_k = f_k \text{ in } \mathbb{R}\times[-T,T]} \|\widetilde{f}_k\|_{N_k} < \infty\}, \end{cases} \tag{16}$$

Next, we will assemble these dyadic function spaces above using a Littlewood-Paley decomposition to obtain the global function spaces. For $u \in C([-T,T]:H^s)$ and $s \geq 0$, we define

$$\|u\|^2_{\mathbf{E}^s(T)} = \|P_{\leq 0}(u(0))\|^2_{H^s} + \sum_{k\geq 1} \sup_{t_k\in[-T,T]} 2^{2sk}\|P_k(u(t_k))\|^2_{E_k}, \tag{17}$$

Finally, the $X_{s,b}$- type control of the solutions, respectively the nonlinearity is achieved using the normed spaces

$$\mathbf{F}^s(T) = \{u \in C([-T,T] : H^s) : \|u\|^2_{\mathbf{F}^s(T)} = \sum_{k=0}^{\infty} 2^{2sk} \|P_k(u)\|^2_{F_k(T)} < \infty\},$$

$$\mathbf{N}^s(T) = \{f \in C([-T,T] : H^s) : \|f\|^2_{\mathbf{N}^s(T)} = \sum_{k=0}^{\infty} 2^{2sk} \|P_k(f)\|^2_{N_k(T)} < \infty\}. \tag{18}$$

**Lemma 3(G-Huo,2013).** *Let* $T \in (0,1]$ *and* $s \in \mathbb{R}_+$, *and* $u \in \mathbf{F}^s(T)$, *then*

$$\sup_{t\in[-T,T]} \|u(t)\|_{H^s} \lesssim \|u\|_{\mathbf{F}^s(T)}. \tag{19}$$

**Lemma 4(Global linear estimate, (G-Huo,2013)).** *Assume* $T \in (0,1]$ *and* $s \in \mathbb{R}_+$, $u, f \in C([-T,T) : H^\infty)$.

$$u_t - iu_{xx} = f \text{ on } \mathbb{R} \times (-T,T). \tag{20}$$

*Then,*

$$\|u\|_{\mathbf{F}^s(T)} \lesssim \|u\|_{\mathbf{E}^s(T)} + \|f\|_{\mathbf{N}^s(T)}. \tag{21}$$

**Remark.** From Lemma 4, we can find that it is difficult to obtain local well-posedness in $H^s$ with $s < 1/2$ by using directly contraction principle. To obtain local well-posedness, we also need energy inequality for equation.

**Theorem 5(Trilinear estimate, (G-Huo,2013)).** *If* $s \geq 1/3$ *and* $T \in (0,1)$, *then*

$$\|u_1 \partial_x \bar{u}_2 u_3\|_{\mathbf{N}^s(T)} \lesssim \|u_1\|_{\mathbf{F}^s(T)} \|u_2\|_{\mathbf{F}^s(T)} \|u_3\|_{\mathbf{F}^s(T)}. \tag{22}$$

**Corollary 6 (Frilinear estimate, (G-Huo,2013)).** *If* $s \geq 1/3$ *and* $T \in (0,1)$, *then*

$$\|u_1 \bar{u}_2 u_3 \bar{u}_4 u_5\|_{\mathbf{N}^s(T)} \lesssim \prod_{j=1}^{5} \|u_j\|_{\mathbf{F}^s(T)}. \tag{23}$$

**Theorem 7 (Energy estimates, (G-Huo,2013)).** *Assume* $s \geq 1/3$ *and* $T \in (0,1)$ *and* $u$ *is a solution of the Cauchy problem (1) in* $C([0,T]; H^\infty)$, *Then*

$$\|u\|^2_{\mathbf{E}^s(T)} \lesssim \|u_0\|^2_{H^s} + \|u\|^4_{\mathbf{F}^s(T)} + \|u\|^6_{\mathbf{F}^s(T)}. \tag{24}$$

Now we turn to the proof of Theorem 1. Note the known result: the Cauchy problem (2) is globally well-posed in $H^1$ assuming the smallness condition on initial data in the $L^2$ norm. In the following proof, we will use this result. First, we prove that if $T \in (0, 1]$ and $u \in C([-T, T] : H^1)$ is a solution of (2) with $u_0 \in H^s$ ($s \geq 1/3$) then

$$\sup_{t \in [-T,T]} \|u(t)\|_{H^1} \lesssim \|u_0\|_{H^1}. \tag{25}$$

By scaling, we may assume

$$\|u_0\|_{H^s} \ll 1. \tag{26}$$

We first use a continuity argument to establish an $\mathbf{F}^s$ bound on $u$ in the interval $[-T,T]$. It follows from Lemma 4, Theorem 5, Corollary 6 and Theorem 7 that for any $T' \in [0,T]$ and $s \geq 1/3$ we have

$$\begin{cases} \|u\|_{\mathbf{F}^s(T')} \lesssim \|u\|_{\mathbf{E}^s(T')} + \|u^2\bar{u}_x\|_{\mathbf{N}^s(T')} + \||u|^4u\|_{\mathbf{N}^s(T')}; \\ \|u^2\bar{u}_x\|_{\mathbf{N}^sT')} \lesssim \|u\|^3_{\mathbf{F}^s(T')}; \\ \||u|^4u\|_{\mathbf{N}^s(T')} \lesssim \|u\|^5_{\mathbf{F}^s(T')}; \\ \|u\|^2_{\mathbf{E}^s(T')} \lesssim \|u_0\|^2_{H^s} + \|u\|^4_{\mathbf{F}^s(T')} + \|u\|^6_{\mathbf{F}^s(T')}. \end{cases} \tag{27}$$

We denote $X(T') = \|u\|_{\mathbf{E}^s(T')} + \|u^2\bar{u}_x\|_{\mathbf{N}^sT')} + \||u|^4u\|_{\mathbf{N}^s(T')}$ and eliminate $\|u\|_{\mathbf{F}^s(T')}$ to obtain

$$X(T')^2 \lesssim \|u_0\|^2_{H^s} + X(T')^4 + X(T')^6 + X(T')^{10} \tag{28}$$

Assuming that $X(T')$ is continuous and satisfies

$$\lim_{T'\to 0} X(T') \lesssim \|u_0\|_{H^s} \tag{29}$$

Using (27), we have

$$\|u\|_{\mathbf{F}^s(T)} \lesssim \|u_0\|_{H^s}. \tag{30}$$

Assume $u_0 \in H^s$ $(s \geq 1/3)$ is fixed,

$$\{\phi_n : n \in \mathbb{Z}_+\} \subseteq H^1 \text{ and } \lim_{n\to\infty} \phi_n = u_0 \text{ in } \ H^s.$$

Let $u_n$ is the solution of the Cauchy problem (2) with initial data $\phi_n$. It suffices to prove that the sequence $u_n \in C([-T,T] : H^1)$ is a Cauchy sequence in $C([-T,T] : H^s)$. By scaling, we may assume

$$\|u_0\|_{H^s} \ll 1, \ \|\phi_n\|_{H^s} \ll 1 \tag{31}$$

It suffices to prove that for any $\delta > 0$ there is $M_\delta$ such that

$$\sup_{t\in[-1,1]} \|u_m(t) - u_n(t)\|_{H^s} \leq \delta \text{ for any } m,n \geq M_\delta. \tag{32}$$

In fact, we use the dyadic energy inequality to obtain (32). This completes the proof Theorem 1.

# I-method and global well-posedness:

**Multilinear expressions.**

If $n \geq 2$ is an even integer, we define a *(spatial) multiplier of order* $n$ to be any function $M_n(\xi_1, \ldots, \xi_n)$ on the hyperplane

$$\Gamma_n := \{(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n : \xi_1 + \ldots + \xi_n = 0\},$$

which we endow with the standard measure $\delta(\xi_1 + \ldots + \xi_n)$, where $\delta$ is the Dirac delta.

If $M_n$ is a $n$-multiplier and $f_1, \ldots, f_n$ are functions on $\mathbb{R}$, we define the $n$*-linear functional* $\Lambda_n(M_n; f_1, \ldots, f_n)$ by

$$\Lambda_n(M_n; f_1, \ldots, f_n) := \int_{\Gamma_n} M_n(\xi_1, \ldots, \xi_n) \prod_{j=1}^{n} \widehat{f}_j(\xi_j).$$

We adopt the notation

$$\Lambda_n(M_n; f) := \Lambda_n(M_n; f, \bar{f}, f, \bar{f}, \dots, f, \bar{f}).$$

Observe that $\Lambda_n(M_n; f)$ is invariant under permutations of the even $\xi_j$ indices, or of the odd $\xi_j$ indices.

In frequency space consider an even $C^\infty$ monotone multiplier $m(\xi)$ taking values in $[0, 1]$ such that

$$m(\xi) := \begin{cases} 1, & \text{if } |\xi| < N, \\ \left(\frac{|\xi|}{N}\right)^{s-1} & \text{if } |\xi| > 2N. \end{cases} \tag{33}$$

Define the multiplier operator $I : H^s \longrightarrow H^1$ such that $\widehat{Iu}(\xi) := m(\xi)\widehat{u}(\xi)$.

Colliander, Keel, Staffilani, Takaoka and Tao (2002) defined an second modified energy

$$E_I^2(u) = -\Lambda_2(m_1\xi_1 m_2\xi_2, u) + \frac{1}{2}\Lambda_4\left(M_4(\xi_1,\xi_2,\xi_3,\xi_4), u\right), \quad (34)$$

where

$$M_4(\xi_1,\xi_2,\xi_3,\xi_4) = \frac{-\sigma_4(\xi_1,\xi_2,\xi_3,\xi_4)}{(\xi_1^2-\xi_2^2+\xi_3^2-\xi_4^2)}. \quad (35)$$

$$\sigma_4(\xi_1,\xi_2,\xi_3,\xi_4) = m_1^2\xi_1^3 + m_2^2\xi_2^3 + m_3^2\xi_3^3 + m_4^2\xi_4^3. \quad (36)$$

Then

$$\frac{d}{dt}E_I^2(u) = \frac{1}{2}\Lambda_6(M_6(\xi_1,\xi_2,\cdots,\xi_6); u(t)) \quad (37)$$

$$M_6(\xi_1, \xi_2, \cdots, \xi_6) = -\frac{i}{36} \sum_{\substack{\{a,c,e\}=\{1,3,5\} \\ \{b,d,f\}=\{2,4,6\}}} \Big( M_4(\xi_{abc}, \xi_d, \xi_e, \xi_f)\xi_{abc} + M_4(\xi_a, \xi_{bcd}, \xi_e, \xi_f)\xi_{bcd} + M_4(\xi_a, \xi_b, \xi_{cde}, \xi_f)\xi_{cde} + M_4(\xi_a, \xi_b, \xi_c, \xi_{def})\xi_{def} \Big). \tag{38}$$

They used the above the second modified energy to show that the Cauchy problem (2) is globally well-posed in $H^s$ with $s > 1/2$.

**Problem:**

Since the Cauchy problem (2) is locally well-posed in $H^s$ with $s \geq 1/3$, which is obtained by Theorem 1. How about of global well-posedness of the Cauchy problem (2) if $s < 1/2$?

But there exists a problem that the second modified energy defined by Colliander, Keel, Staffilani, Takaoka and Tao is not suitable for our results.

**Third modified energy:**

If we define the third modified energy naturally by the 6-linear multiplier $\sigma_6 = \frac{M_6}{\alpha_6}$ , where $\alpha_6 = i(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 + \xi_5^2 - \xi_6^2)$. But $M_6$ can not be estimated by $\alpha_6$. So it is not suitable to define the third modified energy directly.

**Our method:**

Let $\xi_1, \ldots, \xi_n$ be frequencies such that $\xi_1 + \ldots + \xi_n = 0$. Define $N_i := |\xi_i|$, and $N_{ij} := |\xi_{ij}|$. We adopt the notation that

$$1 \leq soprano, alto, tenor, baritone \leq n$$

are the distinct indices such that

$$N_{soprano} \geq N_{alto} \geq N_{tenor} \geq N_{baritone}$$

are the highest, second highest, third highest, and fourth highest values of the frequencies $N_1, \ldots, N_n$ respectively. Since $\xi_1 + \ldots + \xi_n = 0$, we must have $N_{soprano} \sim N_{alto}$. Also, we see that $M_n$ vanishes unless $N_{soprano} \gtrsim N$.

For $M_6(\xi_1, \xi_2, \cdots, \xi_6)$, by symmetry, we assume $N_1 \geq N_3 \geq N_5$, $N_2 \geq N_4 \geq N_6$ in $\Gamma_6$ and $N_5 = N_{fifth}, N_6 = N_{sixth}$. We define sets:

$$\Upsilon_6 = \{(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \in \Gamma_6 : N_{soprano} \geq N_{alto} \gtrsim N\}$$

$$\Upsilon_{61} = \{(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \in \Upsilon_{62} : N_{tenor} \gg N_{baritone}\}.$$

$$\Upsilon_{62} = \{(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \in \Upsilon_{62} : N_{tenor} \sim N_{baritone}\}.$$

$$\Upsilon_{621} = \{(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \in \Upsilon_{622} : N_{12} N_{soprano} \gg N_{tenor}^2\}.$$

$$\Upsilon_{622} = \{(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \in \Upsilon_{622} : N_{12} N_{soprano} \lesssim N_{tenor}^2\}.$$

$$\Upsilon_6 = \Upsilon_{61} \bigcup \Upsilon_{62}, \;\; \Upsilon_{62} = \Upsilon_{621} \bigcup \Upsilon_{622}.$$

**Our third modified energy:**

Then we define the third modified energy

$$E_I^3(u) = E_I^2(u) - \Lambda_6(\frac{\widetilde{M}_6}{\alpha_6}(\xi_1, \xi_2, \cdots, \xi_6)). \tag{39}$$

$$\frac{d}{dt}E_I^3(u) = \frac{1}{2}\Lambda_6(M_6(\xi_1, \xi_2, \cdots, \xi_6)\chi_{\Upsilon_{622}}; u(t)) + \Lambda_8(\widetilde{M}_8(\xi_1, \xi_2, \cdots, \xi_8) \tag{40}$$

where

$$\widetilde{M}_8(\xi_1, \xi_2, \cdots, \xi_8) = -i\sum_{j=1}^{6} X_j^2(\widetilde{M}_6)\xi_{j+1}. \tag{41}$$

$$\chi_{\Upsilon_{622}} = 1, \ if \ (\xi_1, \xi_2, \cdots, \xi_6) \in \Upsilon_{622}; \ otherwise, \ \chi_{\Upsilon_{622}} = 0.$$

**Lemma 8(G-Huo, 2013).** *Let* $1 \leq 2s + \frac{\alpha}{2}$ *and* $0 < \alpha \leq 2s$, $\alpha \leq 2/3$. *For any Schwartz function* $u$, *we have*

$$\left| \int_T^{T+\delta} \Lambda_6(M_6; u(t))\ dt \right| \lesssim \frac{1}{N^{3-\alpha}} \|Iu\|^6_{\mathbf{F}^1([T,T+\delta]\times\mathbb{R})}, (\xi_1, ...\xi_6) \in \Upsilon_{622}. \tag{42}$$

**Lemma 9 (G-Huo, 2013).** *For any Schwartz function* $u$, *we have*

$$\left| \int_T^{T+\delta} \Lambda_8(\widetilde{M}_8; u(t))\ dt \right| \lesssim \frac{1}{N^{3-\alpha}} \|Iu\|^8_{\mathbf{F}^1([T,T+\delta]\times\mathbb{R})}. \tag{43}$$

## Outline of the proof of Theorem 2.

We start by rescaling the solution $u$. Let $\mu > 0$ be chosen later. We observe that $u$ is a solution for (2) if and only if $u_\mu(t,x) = \mu^{-1/2} u(\frac{t}{\mu^2}, \frac{x}{\mu})$ is a solution for (2) with initial data $u_\mu(0) = \mu^{-1/2} u(0, \frac{x}{\mu})$. From Plancherels theorem and a simple computation we see that

$$\|I\partial_x u_\mu(0)\|_{L^2} \leq \frac{N^{1-s}}{\mu^s} \|u(0)\|_{L^2}$$

$$\|Iu_\mu(0)\|_{L^2} \leq \|u(0)\|_{L^2} \ll 1,$$

while we now choose $\mu = N^{\frac{1-s}{s}}$. We see that $\|Iu_\mu(0)\|_{H^1} \lesssim 1$. From Sobolev embedding (or Gagliardo-Nirenberg) we obtain

$$E_I^3(Iu_\mu(0)) \leq C_1.$$

Now suppose inductively that we have a time $T$ such that

$$E_I^3(Iu_\mu(T)) \leq C_1 + C_2 N^{\alpha-3} T.$$

where $C_2 > 0$ is a constant depending on $C_1$ to be chosen later.

We see from Lemmas 8 and 9 that

$$E_I^3(Iu_\mu(T+\delta)) \leq E_I^3(Iu_\mu(T)) + C_3 N^{\alpha-3} T.$$

where $C_3$ depends on $\delta$ and $C_2$.

As a consequence we have thus shown that

$$\|Iu_\mu(T)\|_{H^1} \lesssim 1$$

for all $T \ll N^{3-\alpha}$. From the definition of $I$ this implies that

$$\|u_\mu(T)\|_{H^s} \lesssim C_N,\ T \ll N^{3-\alpha}.$$

Undoing the scaling, this implies that

$$\|u(T)\|_{H^s} \lesssim C_{N,\mu}, \; T \ll N^{3-\alpha}/\mu^2.$$

However, if $s > \frac{2}{5-\alpha}$, then $N^{(5-\alpha)s-2}$ goes to infinity as $N \to \infty$ for all $0 < \alpha \leq 2/3$. W can choose $\alpha$ such that $s$ small.

This show that the Cauchy problem (2) is globally well-posed in $H^s$ with $s \geq \frac{-3+\sqrt{41}}{8}$.

Thank you!