Second Order Approximation of Down and Out European Option Price

Chen,Wei-Da, Hsu,Shan-Chi and Chiang,Tzuu-Shuh

Institute of Mathematics
Academia Sinica,Taipei,Taiwan

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Inroduction and Basic Results

We consider the usual Black-Scholes model with stochastic volatility,

$$
\begin{aligned}
dS_t &= \mu S_t dt + f(y_t) S_t dw_t, \\
dy_t &= 1/\epsilon \cdot m(y_t) dt + 1/\sqrt{\epsilon} \cdot \beta(y_t) d\tilde{z}_t.
\end{aligned}
$$

Here, S_t is the price of a risky asset and y_t will be referred to as the volatility. Also, $\tilde{z}_t$ is correlated to w_t and $\tilde{z}_t = \rho w_t + \sqrt{1-\rho^2} z_t$ where (w_t, z_t) is a two dimensional Brownian motion.

Let $x_t = \log s_t$ and we have

$$\begin{aligned} dx_t &= (\mu - 1/2 \cdot f^2(y_t))dt + f(y_t)dw_t, \\ dy_t &= 1/\epsilon \cdot m(y_t)dt + 1/\sqrt{\epsilon} \cdot \beta(y_t)d\tilde{z}_t. \end{aligned}$$

Under the martingale measure P^* where

$$dP^*/dP = \exp(\int_0^T -(\mu - r)/f(y_t)dw_t - \int_0^T \gamma(y_t)dz_t$$

$$-1/2\int_0^T ((\mu - r)^2/f^2(y_t) + \gamma^2(y_t))dt).$$

we have

$$dx_t = (r - 1/2 \cdot f^2(y_t))dt + f(y_t)dw_t, \text{ and}$$

$$dy_t = (1/\epsilon m(y_t) - 1/\sqrt{\epsilon}\beta(y_t)\Lambda(y_t))dt$$

$$+1/\sqrt{\epsilon} \cdot \beta(y_t)\rho dw_t + \sqrt{1-\rho^2}/\sqrt{\epsilon} \cdot \beta(y_t)dz_t.$$

Suppose a down and out European call option has a strike K and barrier $E < K$ at maturity T. Let $p(t, x, y)$ be its price at time t. Then

$$p^{\epsilon}(t, x, y) = E^{t,x,y}(exp(-r(T-t))h(x_t), \tau > T)$$

where $h(x) = (exp(x) - K)^{+}$ and τ is the hitting time of x_t at the barrier E and satisfies the following pde :

$$L^{\epsilon}p^{\epsilon}(t, x, y) =$$

$$\partial p/\partial t + 1/2 \cdot f^2(y)\partial^2 p/\partial x^2 + 1/\sqrt{\epsilon} \cdot \beta(y)\rho f(y)\partial^2 p/\partial x \partial y$$

$$+1/(2\epsilon) \cdot \beta^2(y)\partial^2 p/\partial y^2 + (r - (1/2)f^2(y))\partial p/\partial x$$

$$-rp + (1/\epsilon \cdot m(y) - 1/\sqrt{\epsilon} \cdot \beta(y)\Lambda(y))\partial p/\partial y = 0.$$

with the terminal condition $p(T, x, y) = h(x)$ and boundary condition $p(t, \log E, y) = 0$.

Collecting terms according the power of ϵ, we introduce the following operators,

$$
\begin{aligned}
L_0 &= 1/2 \cdot \beta^2(y)\partial^2/\partial y^2 + m(y)\partial/\partial y, \\
L_1 &= \beta(y)\rho f(y)\partial^2/\partial y \partial x - \beta(y)\Lambda(y)\partial/\partial y, \quad \text{and} \\
L_2 &= L_{BS}(f^2(y)) \\
&= \partial/\partial t + 1/2 \cdot f^2(y)\partial^2/\partial x^2 + (r - 1/2 \cdot f^2(y))\partial/\partial x - r \cdot .
\end{aligned}
$$

Then $L^\epsilon = 1/\epsilon \cdot L_0 + 1/\sqrt{\epsilon} \cdot L_1 + L_2$ and $p(t,x,y)$ satisfies $(1/\epsilon \cdot L_0 + 1/\sqrt{\epsilon} \cdot L_1 + L_2)p(t,x,y) = 0$ with suitable terminal and boundary conditions.

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The asymptotic expansion we are looking for is follows. Let

- $$\begin{aligned} P(t,x,y) &= P_0(t,x,y) + \sqrt{\epsilon}P_1(t,x,y) + \epsilon P_2(t,x,y) \\ &+\epsilon^{3/2}P_3(t,x,y) + \epsilon^2 P_4(t,x,y) + \ldots \end{aligned}$$

- and

$$\begin{aligned} Q(t,x,y) &= Q_0(\tau,x,y) + \sqrt{\epsilon}Q_1(\tau,x,y) + \epsilon Q_2(\tau,x,y) \\ &+\epsilon^{3/2}Q_3(\tau,x,y) + \epsilon^2 Q_4(\tau,x,y) + \ldots \end{aligned}$$

where $\tau = (T-t)/\epsilon$

- and

$$\begin{aligned} R(t,x,y) &= R_0(t,\xi,y) + \sqrt{\epsilon}R_1(t,\xi,y) + \epsilon R_2(t,\xi,y) \\ &+\epsilon^{3/2}R_3(t,\xi,y) + \epsilon^2 R_4(t,\xi,y) + \ldots \end{aligned}$$

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  where $\xi = (x-E)/\sqrt{\epsilon}$.

- $P(t,x,y)$ will be referred to as the outer expansion,
- $Q(\tau,x,y)$ as the inner expansion
- $R(t,\xi,y)$ the bounday expansion of $p^{\epsilon}(t,x,y)$ respectively.

The results we have is the following.

$$
\begin{aligned}
&|p^{\epsilon}(t,x,y)-(P_0(t,x)+\epsilon^{1/2}P_1(t,x)+\epsilon P_2(t,x,y) \\
&+\epsilon Q_2(\tau,x,y)+\epsilon R_2(t,\xi,y))| \\
&= O(\epsilon^{3/2})
\end{aligned}
$$

The estimate is uniformly over all t,x,y.

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outer expansion

For the outer expansion, we have $(1/\epsilon \cdot L_0 + 1/\sqrt{\epsilon} \cdot L_1 + L_2)(\Sigma_0 \epsilon^{i/2} P_i(t,x,y)) = 0$. Collecting terms according the power of ϵ, we have

$$
\begin{aligned}
& L_0 P_0 = 0, \\
& L_0 P_1 + L_1 P_0 = 0, \\
& L_0 P_2 + L_1 P_1 + L_2 P_0 = 0, \\
& \ldots\ldots\ldots \\
& L_0 P_n + L_1 P_{n-1} + L_2 P_{n-2} = 0.
\end{aligned}
$$

- (order -1), $L_0 P_0 = 0$ is a Poisson equation in y and implies that $P_0(t,x,y) = P_0(t,x)$.
- (order $-1/2$), $L_1 P_0 = 0$ implies $L_0 P_1 = 0$ and thus $P_1(t,x,y) = P_1(t,x)$.
- (order 0), $L_0 P_2 + L_2 P_0 = 0$ is an inhomogeneous equation and has an solution if
- $\overline{L_2 P_0} = \overline{L_2} P_0 = \int (L_2 P_0)(t,x,y)\phi(y)dy = 0$.

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Thus

$$\begin{aligned} L_{BS}(\overline{f^2})P_0 &= 0, \\ P_0(t, E) &= 0, \\ P_0(T, x) &= h(x). \end{aligned}$$

This is the usual down and out European option and has a closed form solution.

Thus $P_2(t,x,y)$ satisfies the following.

$$\begin{aligned} L_0 P_2 &= -L_2 P_0 = -(L_2 P_0 - \overline{L_2 P_0}) \\ &= -1/2(f^2(y) - \overline{f^2})(\partial^2 P_0/\partial x^2 - \partial P_0/\partial x(t,x)) \end{aligned}$$

Hence we have

$$P_2(t,x,y) = -1/2(\partial^2 P_0/\partial x^2 - \partial P_0/\partial x(t,x))\psi(y) + A_1(t,x),$$

where $\psi(y)$ is the solution of

$$\begin{aligned} L_0\psi(y) &= -1/2(f^2(y) - \overline{f^2}) \\ \overline{\psi} &= 0 \end{aligned}$$

and $A_1(t,x)$ is to be determined.

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Consider the next order $\sqrt{\epsilon}$,

$$L_0 P_3 + L_1 P_2 + L_2 P_1 = 0.$$

Hence P_3 has a solution if $\overline{L_1 P_2 + L_2 P_1} = 0$. We need

$$\overline{L_2} P_1 = L_{BS}(\overline{f^2}) P_1 = -\overline{L_1 P_2}.$$

Since $\overline{L_1 P_2}$ is independent of $A_1(t, x)$, we conclude that $P_1(t, x)$ solves the following.

$$L_{BS}(\overline{f^2}) \quad \begin{array}{l} P_1(t, x) = H(t, x), \\ P_1(T, x) = 0, \quad \text{and} \\ P_1(t, E) = 0 \end{array}$$

where $H(t, x) = -\overline{L_1 P_2}$ is a known function.
Thus $P_1(t, x)$ is determined and has a closed form solution using method of images.

- We can not find $A_1(T,x)$ so that $P_2(T,x,y)=0$ for every x,y.
- This is the reason to have the inner expansion $Q_i(\tau,x,y), \tau=(T-t)/\epsilon$.
- We need $P_2(T,x,y)+Q_2(0,x,y)=0$ to match the terminal condition.
- When t is away from T, we have $Q_2(\tau,x,y)\rightarrow 0$ as $\tau\rightarrow\infty$.

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For the inner expansion, we have
$(1/\epsilon \cdot L_0 + 1/\sqrt{\epsilon} \cdot L_1 + L_2)(\Sigma_0 \epsilon^{i/2} Q_i((T-t)/\epsilon, x, y)) = 0$. Let

$$
\begin{aligned}
\tilde{L}_0 &= -\partial/\partial\tau + 1/2 \cdot \beta^2(y)\partial^2/\partial y^2 + m(y)\partial/\partial y, \\
\tilde{L}_1 &= \beta(y)\rho f(y)\partial^2/\partial y\partial x - \beta(y)\Lambda(y)\partial/\partial y = L_1, \quad \text{and} \\
\tilde{L}_2 &= 1/2 \cdot f^2(y)\partial^2/\partial x^2 + (r - 1/2 \cdot f^2(y))\partial/\partial x - r \cdot .
\end{aligned}
$$

We have

$$
\tilde{L}(\Sigma_0 \epsilon^{i/2} Q_i(\tau, x, y)) = (\epsilon \tilde{L}_0 + \sqrt{\epsilon}\tilde{L}_1 + \tilde{L}_2)(\Sigma_0 \epsilon^{i/2} Q_i(\tau, x, y)) = 0
$$

where $\tau = (T-t)/\epsilon$

- For the terms of ϵ^{-1}, we have $\tilde{L}_0 Q_0(\tau, x, y) = 0$ with initial condition $Q_0(0, x, y) = 0$,
- hence $Q_0(\tau, x, y) = 0$.
- Simiarly, $Q_1(\tau, x, y) = 0$.
- For the terms of ϵ^0, we have $\tilde{L}_0 Q_2(\tau, x, y) + \tilde{L}_1 Q_1 + \tilde{L}_2 Q_0 = \tilde{L}_0 Q_2(\tau, x, y) = 0$.

We have the initial condition
$Q_2(0, x, y) = -P_2(T, x, y) = 1/2(h''(x) - h'(x))\psi(y) - A_1(T, x)$. Since $Q_2(\tau, x, y) \to -\overline{P_2(T, x, y)} = -\overline{P_2(T, x, \cdot)}$ as $\tau \to \infty$ by the ergodic theorem, we have $\overline{P_2(T, x, \cdot)} = 0 = A_1(T, x)$.

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Thus $Q_2(\tau, x, y) = 1/2(h''(x) - h'(x))\phi(\tau, y)$ where $\phi(\tau, y)$ solves

$$\begin{aligned} \tilde{L}_0\phi(\tau, y) &= 0 \\ \phi(0, y) &= \Phi(y). \end{aligned}$$

Note that if we require $h''(x) - h'(x) = 0$, which is the case when $E \leq K$, then $Q_2(\tau, E, y) = 0$.

Return to the equation of $P_3(t,x,y)$. Since $L_0P_3 + L_1P_2 + L_2P_1 = 0$ and $P_1 + P_2$ is known, we write

$$P_3(t,x,y) = \tilde{P}_3(t,x,y) + A_2(t,x)$$

where $\tilde{P}_3$ solves

$$L_0\tilde{P}_3 = -L_1P_2 - L_2P_1$$

with $\overline{\tilde{P}_3} = 0$. Now only $A_2(t,x)$ remains to be determined.

Consider the next order of ϵ, we have $L_0 P_4 + L_1 P_3 + L_2 P_2 = 0$.
For P_4 to have a solution, we need $\overline{L_2 P_2} = \overline{-L_1 P_3}$.
This leads to $L_{BS}(\overline{f^2}) A_1(t,x) + G(t,x) = \overline{-L_1 P_3}$ where $G(t,x)$ is a known function.
Thus $A_1(t,x)$ is determined and so is $P_2(t,x,y)$.

Consider the next order of ϵ, we have $L_0P_4 + L_1P_3 + L_2P_2 = 0$. For P_4 to have a solution, we need $\overline{L_2P_2} = \overline{-L_1P_3}$.

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Thus $A_1(t,x)$ is determined and so is $P_2(t,x,y)$.

It is not true that $P_2(t, E, y) = 0$ and we need to consider the boundary expansion. Let

$$\begin{aligned}
\hat{L}_0 &= 1/2 \cdot \beta^2(y)\partial^2/\partial y^2 + 1/2 f^2(y)\partial^2/\partial\xi^2 + m(y)\partial/\partial y \\
&\quad + \rho\beta(y)f(y)\partial^2/\partial\xi\partial y \\
\hat{L}_1 &= (r - 1/2 \cdot f^2(y))\partial/\partial\xi - \beta(y)\Lambda(y)\partial/\partial y \quad \text{and} \\
\hat{L}_2 &= \partial/\partial t - r \cdot .
\end{aligned}$$

Then $(1/\epsilon \cdot L_0 + 1/\sqrt{\epsilon} \cdot L_1 + L_2)(\Sigma_0 \epsilon^{i/2} R_i(t, (x - E)/\sqrt{\epsilon}, y)) = 0$ becomes

$$\hat{L}(\Sigma_0 \epsilon^{i/2} R_i(t, \xi, y)) = (\epsilon\hat{L}_0 + 1/\sqrt{\epsilon}\hat{L}_1 + \hat{L}_2)(\Sigma_0 \epsilon^{i/2} R_i(t, \xi, y)) = 0$$

where $\xi = (x - E)/\sqrt{\epsilon}$.

Consider the term of order ϵ^{-1},

$$\hat{L}_0 R_0(t, \xi, y) = 0$$

and the boundary condition $R_0(t, 0, y) = 0$. Let

$$\Theta = \begin{pmatrix} \sqrt{1-\nu^2} f(y) & \nu f(y) \\ \nu\beta(y) & \sqrt{1-\nu^2}\beta(y) \end{pmatrix}$$

then we have

$$\Theta\Theta^t = \begin{pmatrix} f^2(y) & 2\nu\sqrt{1-\nu^2}\beta(y)f(y) \\ 2\nu\sqrt{1-\nu^2}\beta(y)f(y) & \beta^2(y) \end{pmatrix}$$

For any $-1 \leq \rho \leq 1$, there is a unique ν such that $\nu\sqrt{1-\nu^2} = 1/2\rho$.

Obviously, $\hat{L}_0(\xi, y)$ is the generator of a process (ξ_t, y_t). Thus $R_0(\xi, y) = 0$. (A Poisson equation with 0 boundary condition.) Similarly, $R_1(\xi, y) = 0$.

The first non-trivial term is R_2 because $R_2(t, 0, y)$ is not 0. We have

$$\hat{L}_0 R_2(t, \xi, y) = 0$$
$$R_2(t, 0, y) = -P_2(t, E, y).$$

Thus

$$R_2(t, \xi, y) = E^{\xi,y}(-P_2(t, E, y_\tau))$$
$$= 1/2(\partial^2 P_0/\partial x^2 - \partial P_0/\partial x(t, E))E^{\xi,y}(\psi(y_\tau)) - A_1(t, E)$$

where $\tau = \inf(s, \xi_s = 0)$.

The first non-trivial term is R_2 because $R_2(t,0,y)$ is not 0. We have

$$\begin{aligned} &\hat{L}_0 R_2(t,\xi,y) = 0 \\ &R_2(t,0,y) = -P_2(t,E,y). \end{aligned}$$

Thus

$$\begin{aligned} &R_2(t,\xi,y) = E^{\xi,y}(-P_2(t,E,y_\tau)) \\ &= 1/2(\partial^2 P_0/\partial x^2 - \partial P_0/\partial x(t,E))E^{\xi,y}(\psi(y_\tau)) - A_1(t,E) \end{aligned}$$

where $\tau = \inf(s, \xi_s = 0)$.

Obviously, $R_2(t,0,y) + P_2(t,E,y) = 0$ and

$$
\begin{aligned}
& P_2(T,\xi,y) \\
&= 1/2(\partial^2 P_0/\partial x^2 - \partial P_0/\partial x(T,E))E^{\xi,y}(\psi(y_\tau)) - A_1(T,E) \\
&= 1/2(h''(E) - h'(E))\psi(y) - A_1(T,E) = 0
\end{aligned}
$$

if $h''(E) - h'(E) = 0$. Also, $Q_2(\tau,E,y) = 0$ if $h''(E) - h'(E) = 0$.

Let

$$\Theta = \begin{pmatrix} \sqrt{1-\nu^2} f(y) & \nu f(y) \\ \nu\beta(y) & \sqrt{1-\nu^2}\beta(y) \end{pmatrix}$$

Then we have

$$\Theta\Theta^t = \begin{pmatrix} f^2(y) & 2\nu\sqrt{1-\nu^2}\beta(y)f(y) \\ 2\nu\sqrt{1-\nu^2}\beta(y)f(y) & \beta^2(y) \end{pmatrix}$$

We need to show that

$$
\begin{aligned}
&|P^\epsilon(t,x,y) - (P_0(t,x) + \sqrt{\epsilon}P_1(t,x) + \epsilon P_2(t,x,y)) \\
&-\epsilon Q_2(T - t/\epsilon, x, y) - R_2(t, x - E/\sqrt{\epsilon}, y)| \\
&= O(\epsilon^{3/2}).
\end{aligned}
$$

The estimate is uniform over t, x, y. Let $h''(E) = h'(E)$. Then we have the following terminal and boundary conditions :

$$P_0(T,x)+\sqrt{\epsilon}P_1(T,x)+\epsilon P_2(T,x,y)$$
$$+\epsilon Q_2(0,x,y)+\epsilon R_2(T,x-E/\sqrt{\epsilon},y)=h(x) \text{ and}$$

$$P_0(t,E)+\sqrt{\epsilon}P_1(t,E)+\epsilon P_2(t,E,y)$$
$$+\epsilon Q_2(T-t/\epsilon,x,y)+\epsilon R_2(t,0,y)=0.$$

Let

$$
\begin{aligned}
Z_4(t,x,y) \quad &= P_0(t,x) + \sqrt{\epsilon} P_1(t,x) + \epsilon P_2(t,x,y) \\
&+ \epsilon^{3/2} P_3(t,x,y) + \epsilon^2 P_4(t,x,y) \\
&+ \epsilon Q_2(\tau,x,y) + \epsilon^{3/2} Q_3(\tau,x,y) + \epsilon^2 Q_4(\tau,x,y) \\
&+ \epsilon R_2(t,\xi,y) + \epsilon^{3/2} R_3(t,\xi,y) + \epsilon^2 R_4(t,\xi,y) \\
&- p(t,x,y).
\end{aligned}
$$

Then Z_4 satisfies the following equation:

$$LZ_4(t,x,y) = \epsilon^{3/2} F^{\epsilon}(t,x,y)$$

with terminal and boundary conditions

$$Z_4(T,x,y) = \epsilon^{3/2} G^{\epsilon}(x,y)$$

and

$$Z_4(t,E,y) = \epsilon^{3/2} H^{\epsilon}(t,y).$$

where $|F^{\epsilon}(t,x,y)|, |G^{\epsilon}(x,y)|$ and $|H^{\epsilon}(t,y)|$ are of $O(1)$ uniformly in t, x and y.

Feynman-Kac formula then implies that

$$sup_{t,x,y}|Z_4(t,x,y)| = O(\epsilon^{3/2}).$$

Thus

$$\begin{aligned} p(t,x,y) \quad & -(P_0(t,x) + \sqrt{\epsilon}P_1(t,x) + \epsilon P_2(t,x,y) \\ & +\epsilon Q_2(\tau,x,y) + \epsilon R_2(t,\xi,y)) \\ & = Z_4(t,x,y) - \epsilon^{3/2}R_3(t,\xi,y) - \epsilon^2 R_4(t,\xi,y) \\ & -\epsilon^{3/2}P_3(t,x,y) - \epsilon^2 P_4(t,x,y) \\ & -\epsilon^{3/2}Q_3(\tau,x,y) + \epsilon^2 Q_4(\tau,x,y) \\ & = O(\epsilon^{3/2}). \end{aligned}$$

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