On the quasi- interpolation

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Outline

Basic Concepts

Applied Mathematics = Application $\overset{\text{Interface}}{\Longleftarrow\!=\!=\!\Longrightarrow}$ Mathematics.

Media of the Interaction $\Longrightarrow$ Data.

Basic Problem: Data $\xRightarrow{\text{Representation}}$ Function.

After Representation, Mathematician Works on the Function.

Mathematical Modeling = Working on Function, and interpret it as

Physical Law. Kepler's law was established in this way.

Data $\xRightarrow{\text{Representation}}$ Function

To approximate the function, requires a nested function space.

e.g. $\cdots \subset V_n \subset V_{n+1} \subset \cdots$.

for example $V_n = P_n$, the spline, the wavelets, etc.

Find $f_n(x) \in V_n$ based on the Data $\{f(x_j)\}_{j=0}^{N}$, such that

$\|f_n(x) - f(x)\| = \min.$ (If Euclidean or Hilbert, then projection)

Interpolation, distance of data

e.g. $V_n = P_n$ the polynomial of degree less or equal to $n-1$.

Existence: If $n \geq N$, the interpolation are successfully.

$n = N$ uniquely solvable. $n > N$ no uniqueness, however

we can find a solution with a punish of $\|f_n(x)\|_{L_0} = \min$ to get a shortest representation, which is very popular recently.

Usually n very large (too large!), especially for multi variate problem, require to solve a large scaled linear system of equation, or even with a searching process of non linear programming.

Scattered Data and linear functional Data

The knots or data points $\{x_j\}_{j=0}^N$ are usually scattered.

A lot of the cases in the application, the data are even linear functional.(e.g. by remote sensing, seismic, etc).

Denote D to be differential operator. One can measure $P(D)f$ at the knots $\{x_j\}_{j=0}^N$, where P is assumed to be a polynomial.

Space and Basis

Key feature of the function representation: Using less basic functions to represent or approximate target (complicate) function.

Function is a (or can be constructed by) linear combination of basis.

What is basis?

The smallest set, which can be combined to represent functions.

How many basis for 100-variate parabolic function space,

$\longrightarrow$ Catastrophe of the space-dimension.

Generator of the physical world, fundamental particle

What is the substance? Ask Physicists.

Substance is constructed (linear combined) by Atoms or Elements.

e.g. Water=2H+O, Salt=Na+Cl.

Basis $\sim$ Atoms or Elements.

Atoms or elements are constructed by more fundamental particles!

If in function space exists such more basic function than basis?

Generator, Definition

Definition

For a given function space, if there exists a function $G(x)$, such that the shifts and linear combination can generate the function space, then $G(x)$ is called the **generator** *of the function space.*

Remark

Generator *is a more basic function than the basis of the function space, which contains all the DNA of the function space. Require only shifts (copy it self) and liner combination. This phenomena appears often in Biology too (protein, cell, life $\rightarrow$ cell, life, society).*

Most of common function space exist generator

- x^n, Polynomial.
- $\sin(2x) - 2\sin(x)$, Trigonometric Polynomial.
- $2e^x - e^{2x}$, Exponential Polynomial.
- $|x|^{2k+1}$, Polynomial Splines.
- $|x|^{2k+1}e^x \sin^2 x$, Tschebycheffian Splines.
- e^{-x^2}, Gaussian.
- $\sqrt{c^2 + x^2}$, Multi Quadratics.

In the application, the Engineer don't learned a lot of function. Above are almost all kinds of the function, which have been learned by Engineer. (This is perhaps the reason, why the spline wavelets is mostly used in wavelets application)

Generator, which generates finite dimensional space

More than $n+1$ shifts of x^n is linearly dependent.

Theorem

Necessary and Sufficient Condition Function which only generate finite dimensional space must be a solution of linear ordinary differential equation with constant coefficients.

For multivariate cases, the condition is necessary but not sufficient.

Examples: D=differential operator, I=identical operator

- $D^n f(x) = 0$, Polynomial of degree $n-1$
- $(D^2 + c^2 I) f(x) = 0$, Trigonometry polynomial
- $(D^2 - c^2 I) f(x) = 0$, Exponential polynomial
- $D^n (D^4 - c^4 I) f(x) = 0$, Algebra of above functions
- $P(D) f(x) = 0$, If $P(\lambda) = 0$ possesses roots λ_j, then $\{\exp(\lambda_j x)\}$ is basis, $\sum \exp(\lambda_j x)$ is a generator.

Remark

Contain almost all the function, which can be supplied to the application. Using rational form we can get more other function space, $\tan(x) = \{\sin(x), \cos(x)\}$ *functions pair of above function.*

Standard Generator, Finite Dimensional Space

Theorem

For the solution space of $P(D)G(x) = 0$, if $G(x)$ satisfies

$$G^{(k)}(0) = \delta_{k,n-1},$$

we can prove: $G(x)$ is a generator of the solution space and is called the **standard generator** *for the solutions space.*

Theorem

The function $G(x)$ is the first coefficient of interpolatory polynomial that the value at λ_j is $e^{\lambda_j x}$, which can be represented in the form of divided difference, an explicit representation of the generator.

$$G(x) = [\lambda_1, \cdots, \lambda_n] e^{\lambda x}$$

Increasing the Capacity of the Generator

The approximation capacity is limited, if the generator can only generate finite dimensional function space.

How can we construct a new generator based on the standard generator of the solution space of the ordinary differential equations, such that it can generate an infinite dimensional space and can approximate (represent) almost all the functions?

Ask the Biologist!

Infinite Dimensional Space

Bio-multiformity —- the reason, a new species appear, will happen via **Gene mutation**.
Based on the generator $x \to |x|$, Euclidian hat can be construct:

$$\Lambda_j(x) = \frac{(|x - x_{j+1}| - |x - x_J|)}{2(x_{j+1} - x_J)} - \frac{(|x - x_j| - |x - x_{j-1}|)}{2(x_j - x_{j-1})}$$

Any standard generator with Gene mutation:

$$G(x) := \frac{1}{2}\mathrm{sign}(x)G(x)$$

will generate an infinite dimensional space (Tschebycheffian spline: piecewise function of algebra of polynomial, trigonometric polynomial and exponential polynomial) and can approximate (represent) almost any function.

For Tschebycheffian spline generated by such generator.

Parallel works such as the dual basis, the interpolation and the approximation, B-spline form, Energy minimization, subdivisions algorithm, wavelets decomposition have been done (W).

Multivariant Problem

Radial generator: $\phi : R_+ \to R, \Phi(x) = \phi(\|x\|)$

Radial basis space $\{\Phi(x - x_j)\}$ simple for multi variant problem.

We can construct the dual basis for given Hilbert, Sobolev norm.

We can design Measurement or Norm to form a orthogonal basis.

By using the concept of Reproducing Kernel Hilbert space.

Existence: Conversely, if a function space equipped with an inner product, then the corresponding reproducing kernel (=generator) can be computed too.

Shift invariant Reproducing Hilbert space

Any shift invariant Hilbert space possesses a reproducing generator

$$\phi(x-y) = \sum b_j(x)\overline{b_j(y)},$$

where $\{b_j(x)\}$ is any orthonormal basis.

that $f(x) =< f(\cdot), \phi(x-\cdot) >$.

Key problem is to find simple mathematical representation of the generator for using in application.

More Consideration

Today we will not go into the details of the construction of the generator, however to construct the approximation by generator.

We can use interpolation or least square as approximation.

Interpolation and least square require to solve a linear system of equations.

Can we save the step of solving linear system of equations?

Yes! Interpolation $\Rightarrow$ Quasi-interpolation.

Examples of classical Quasi-interpolation

e.g. Bernstein's Polynomial $f(x) \sim \sum f(j/n) B_j^n(x),\ 0 \leq j \leq n$.

e.g. B-spline in CAGD $f(x) \sim \sum f(jh) B(\frac{x}{h} - j)$.

e.g. Shannon Sampling in Signal Processing $f(x) = \sum f(jh) \mathrm{sinc}(\frac{x}{h} - j)$ for Bandlimit function $f(x)$ (Mobil phone, TV).

Bernstein, B-spline and sinc function are some pre-given functions. They yield the solution directly, do not require to solve any linear system of equations.

Introducing: MQ:$\sqrt{c^2 + x^2}$, generator by Hardy in Boeing Co..

Related Works on Multiquadric Quasi-interpolation

- Beatson and Powell constructed 3 schemes of Multiquadric quasi-interpolation (which is applied in the movie "The Lord of the Rings III") .
- W. and Schaback generalized Beatson-Powell result and constructed a MQ-QI scheme. Showed it is shape preserving and constant reproducing and gave the error estimates.
- Beatson and Dyn, gave theoretical discussion on the MQ-QI.
- Buhmann and Powell, Error estimates for derivative.
- Ling used multilevel method to improve the accuracy of W-Schaback's scheme.
- Chen and W. constructed the linear reproducing scheme and periodic scheme.
- Feng constructed a quadric reproducing scheme.
- W. & Zhang, W. & Xu and Feng presented schemes for high order polynomial reproducing.

Quasi-interpolation for Burger's Equation

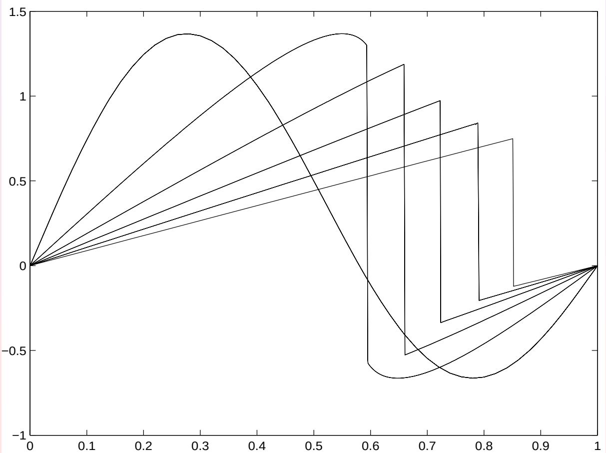


Figure: The Shock Wave

Using Multiquadric Quasi-interpolation to Solve PDEs

- Hon & W., the cracks for boundary testing, the boundary layer problem.
- W. and Chen & W., Nonlinear PDEs such as Burgers equation.
- Ma & W.,Sine-Gordon Equation, as well as to detect the coefficients in inverse problem.

Schoenberg's Model and Strang-Fix Conditions

Schoenberg's model (1951)

$$f^*(x) = \sum f(jh)\phi(\frac{x}{h} - j)$$

Related works: Shift invariant space.

The scheme is an approximation of order k,

if and only if

Strang-Fix conditions: If the Fourier transform of ϕ satisfying

$$\begin{aligned} \hat{\phi}(\omega) &= 1 + \mathcal{O}(\omega^k) \\ \hat{\phi}(\omega + 2\pi j) &= \mathcal{O}(\omega^k) \end{aligned}$$

Multivariate Problems

Jia etc: In shift invariant space $\Leftrightarrow$ exists a function (now be called generator) satisfies Strang-Fix conditions.

W. generalized Strang-Fix conditions $\to$ Asymptotical Strang-Fix conditions and constructed new quasi-interpolation model

$$f^*(x) = \sum f(jh)h^p \phi(\frac{x}{h^q} - jh^p), \quad p + q = 1$$

Approximation order depends only to the continuity of ϕ, furthermore, the optimal parameter p was found.

Quasi-interpolation for Multivariate Scattered Data

Question: How about radial functions generator?

Question: How about the scattered data?

Answer: There is no radial function and its finite linear combination

of shift satisfying Strang- Fix conditions(W.).

However

Quasi-interpolation for Multivariate Scattered Data

Theorem

(W.) Given any function ϕ, if $\int \phi(x)dx \neq 0$, then we can find Δ_j (Coefficients of Gaussian Quadrature), that

$$f^*(x) = \sum \frac{f(x_j)}{h}\phi(\frac{x - x_j}{h})\Delta_j$$

is an approximation. Δ_j depend only on the geometry of the distribution of the data points x_j locally, the approximation order depends on the continuity of ϕ, f and approximation order of the Gaussian Quadrature Δ_j.

MQ Quasi-interpolation for high order derivative

Multiquadric quasi-interpolation:

$$\phi(x) = \sqrt{c^2 + \|x\|^2} \quad \phi_j(x) = \phi(x - x_j)$$

$$\psi_j(x) = \frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})}$$

$$f^*(x) = \sum f(x_j)\psi_j(x)$$

High Order Derivatives of Multiquadric Quasi-interpolation

To solve partial differential equation
→ require to approximate the high order derivatives

Compare to the high order approximation, usually

$$\|f^{*(k)}(x) - f^{(k)}(x)\| < \mathcal{O}(h^{n-k}) \tag{1}$$

However we require an almost same approximation order for the function as well as for the derivative.

Multiquadric Quasi-interpolation, Approximation Order

Multiquadric quasi-interpolation scheme:

$$(\mathcal{L}f)(x) = \sum_{j=-\infty}^{\infty} f(x_j)\psi_j(x), \tag{2}$$

where $\psi_j(x)$ are defined as above.

Theorem

Buhmann, Powell, Schaback and W. Let $f(x)$ be twice differentiable, such that $\|f'(x)\|_\infty$ and $\|f''(x)\|_\infty$ are bounded. Then the inequalities

$$\|(\mathcal{L}f)(x) - f(x)\|_\infty \leq \mathcal{O}(h^2)$$

and

$$\|(\mathcal{L}f)'(x) - f'(x)\|_\infty \leq \mathcal{O}(h)$$

hold, provided that c is small enough.

Better result of approximation to derivative

Theorem

If $f(x) \in C^{(k+1)}(R)$ *and* $f^{(j)}(x)$ *is bounded by a polynomial of degree* $k+2-j$*, then*

$$|(\mathcal{L}f)^{(k)}(x) - f^{(k)}(x)| \leq \mathcal{O}(h^{\frac{2}{k+1}})$$

holds, provide that $c = h^{\frac{1}{k+1}}$ *is choosing properly, since* $E_{rror} = \mathcal{O}(c^2 + h/c^{k-1})$.

Corollary

Corollary

If the shape parameter $c = h^{\frac{1}{k+1}}$ *is optimal selected as above, then for any* $l \leq k$*,*

$$|f^{(l)}(x) - (\mathcal{L}f)^{(l)}(x)| \leq \mathcal{O}(h^{\frac{2}{k+1}});$$

and

$$|f^{(k+1)}(x) - (\mathcal{L}f)^{(k+1)}(x)| \leq \mathcal{O}(h^{\frac{1}{k+1}}).$$

Stability for the Data with Noise

Actually, the function values $\{f(x_j); j = 0, 1, \cdots, N\}$ usually can not be obtained exactly. Usually, $\{f^*(x_j) = f(x_j) + \xi_j\}$, where $\{\xi_j\}$ are white noise.

Assume

$$\mathbf{E}\xi_j = 0, \qquad j = 0, \cdots, n \tag{3}$$

and

$$\mathbf{E}\xi_j\xi_k = \begin{cases} \sigma^2, & j = k, \\ 0, & j \neq k. \end{cases} \tag{4}$$

Stability Analysis for Divided Difference Method

In approximation theory, we know the divided differences is the first coefficient of the interpolatory polynomial, by using the Lagrange interpolation formula, we can get an explicit symmetric representation of the divided differences that

$$[x_0, x_1, \cdots, x_n]f = \sum_{k=0}^{n} \frac{f(x_k)}{\prod_{j \neq k} (x_k - x_j)}. \tag{5}$$

Roughly speaking, divided difference is a local approximation by polynomial interpolation.

Stability of Divided Difference Method

Theorem

If $\{x_j; j = 0, \cdots, n\}$ *are uniformly distributed and* H *denotes the step length, and if* $|x - \xi| \sim \mathcal{O}(H)$, *then*

$$E[(\mathcal{D}f^*)^{(n)}(x) - f^{(n)}(x)]^2 \leq \mathcal{O}\left(\frac{\sigma^2}{H^{2n}}\right) + \mathcal{O}(H^2),$$

where $\mathcal{D}$ *is divided difference operator on* $[x_0, x_1, \cdots, x_n]$.

Remark for Divided Difference Method

Remark

To make sure that the error caused by the instability of the divided difference (the first term) is dominated by the theoretical error (the second term), σ^2 should satisfy

$$\sigma_{\mathcal{D}}^2 \leq \mathcal{O}(H^{2n+2}).$$

This means for a 64-bit computer, the computer possesses only about 20 significant decimal digit ($\sigma_{\mathcal{D}}^2 > 10^{-40}$), then we can only use the divided difference of order 5 to data density of $H = 10^{-2}$ and the result is valid only with 2 decimal digit.

Stability of Multiquadric Quasi-interpolation

Theorem

For the multiquadric quasi-interpolation operator $\mathcal{L}$, *let* $c = \mathcal{O}(h^{\frac{1}{n+1}})$, *then*

$$\mathbf{E}[(\mathcal{L}f^*)^{(n)}(x) - f^{(n)}(x)]^2 \leq \mathcal{O}(\frac{\sigma^2}{h^{\frac{n}{n+1}}}) + \mathcal{O}(h^{\frac{4}{n+1}})$$

holds.

Remark for Multiquadric Quasi-interpolation

Remark

By setting $h^{\frac{2}{n+1}} = H$ to obtain the same theoretical approximation order as divided differences, it requires only that

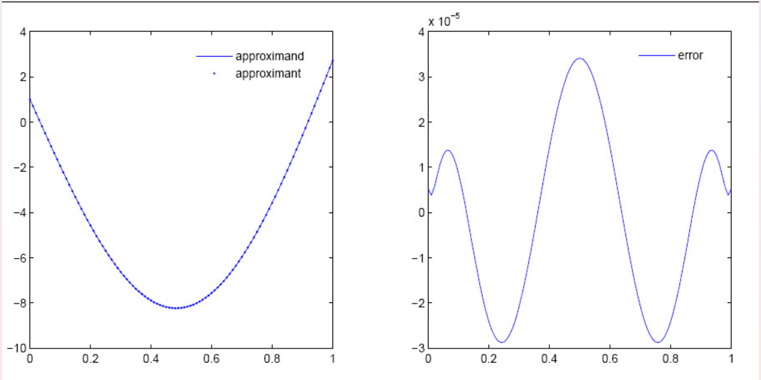
$$\sigma_{\mathcal{L}}^2 \leq \mathcal{O}(H^{\frac{n}{2}+2}).$$

That means multiquadric quasi-interpolation scheme is much stabler than divided difference

$$\sigma_{\mathcal{D}}^2 << \sigma_{\mathcal{L}}^2.$$

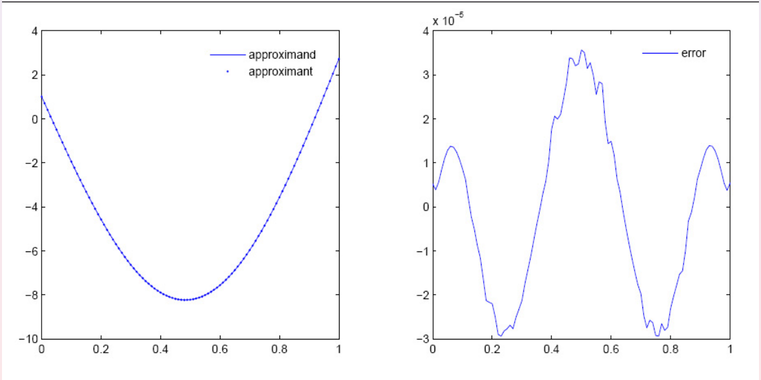
Compared with the example for the divided difference, when it possesses only 2 significant decimal digit, the Multiquadric quasi-interpolation will possesses 8 significant decimal digit.

Numerical Examples



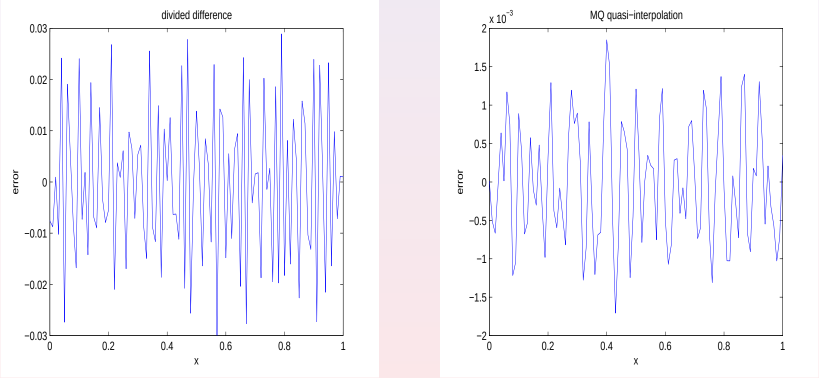
: Plot of function $(\mathcal{L}f)''(x)$ and $f''(x)$ for a uniform data points ($h = 0.01$) (left) and the error function(right) using Multiquadric quasi-interpolation with $c = 0.1h^{1/3}$.

Numerical Examples



: Plot of function $(\mathcal{L}f)''(x)$ and $f''(x)$ for a scattered data points $(h = 0.01)$ (left) and the error function(right) using Multiquadric quasi-interpolation with $c = 0.1h^{1/3}$.

Data with noise and in [0.1,0.9]



: Error by Divided difference and Multiquadric quasi-interpolation with noise

Some Remarks

From the above Figures of the errors, we observed that the multiquadric quasi-interpolation possesses lower frequency and smaller amplitude of the error oscillation than divided difference.

Quasi interpolation for functional data

We have already

1. generalized the classic quasi- interpolation to scattered data
2. gotten a simple quasi- interpolation scheme with good approximation order even for high order derivative.
3. Applied the scheme to developing equation (parabolic and hyperbolic even nonlinear)

Recall the scheme of Hermitian Birkhhoffian interpolation.

(W.) Hermite-Birkhoff interpolation of scattered data by radial basis functions., Approx. Theory Appl. 8 (1992), no. 2, 1–10

Now we will generalize the scheme for linear functional data, furthermore especially use the scheme directly for numerical solution of PDE.

Quasi interpolation for functional data

$$P(D)u(x) = f(x) \qquad x \in \Omega$$
$$Q(D)u(x) = g(x) \qquad x \in \partial\Omega$$

Riesz's representation of the equation, (generalized function P, Q)

$$\int_\Omega P(x,t)u(t) = f(x) \qquad x \in \Omega$$
$$\int_{\partial\Omega} Q(x,t)u(t) = g(x) \qquad x \in \partial\Omega$$

If the equation is well defined, then exists generalized function λ, μ

$$u(x) = \int_\Omega \lambda(x,t)f(t)dt + \int_{\partial\Omega} \mu(x,s)g(s)ds$$

Therefor we have a quasi interpolation scheme by using Riemannian summation to the integral (non Riemannian Integrable)

$$u(x) \sim \sum_j \lambda(x,x_j)f(x_j)\Delta_j + \sum_k \mu(x,y_k)g(y_k)\Delta_k$$

Quasi interpolation for functional data

Condition of the generalized function

$$\int_{\Omega} \lambda(x,t)P(t,y)dt + \int_{\partial\Omega} \mu(x,t)Q(t,y)dt = \delta(x-y)$$

and the superpositions condition of

$$\int_{\Omega} \lambda(x,t)Q(t,y)dt = 0$$

$$\int_{\partial\Omega} \mu(x,t)P(t,y)dt = 0$$

Quasi interpolation for functional data

Usually the discretion of the integral by Riemannian summation can not be succeeded, because the integrand is a generalized function. Therefor we use quasi interpolation to the left side at first to get

$$u(x) \sim \int_\Omega \lambda(x,t) \sum \phi(t - x_j) f(x_j) dt + \int_{\partial\Omega} \mu(x,s) \sum \psi(s - y_k) g(y_k) ds$$

and Redefine

$$\lambda_j(x) := \int_\Omega \lambda(x,t)\phi(t - x_j)dt,$$

$$\mu_k(x) := \int_{\partial\Omega} \mu(x,t)\psi(t - y_k)dt$$

Functional data from partial differential equation

Then we have a quasi interpolation's scheme for the functional data come from some differential equation

$$u(x) \sim \sum_j \lambda_j(x) f(x_j) + \sum_k \mu_k(x) g(y_k),$$

where the two part solve the homogenous equation for differential operator and the boundary condition respectively.
The key feature is to find the λ_j, μ_j with some simple mathematical representation. (Hope λ_j, μ_j are simple and can be pre-given)

Quasi interpolation for solving PDE

Result:

From the discussion of quasi interpolation for scattered data,

we require only to find some function

$\phi(x)$, that $\int_\Omega P(D)\phi(x)dx = 1$,

$\psi(x)$, that $P(D)\psi(s) = 0$, and $\int_{\partial\Omega} Q(D)\psi(x)dx = 1$,

Then let $u_1(x) = \sum f(x_j)\frac{1}{h^p}\phi(\frac{x-x_j}{h^p})\Delta_j$,

that $P(D)u_1(x) \sim f(x)$,

Quasi interpolation for solving PDE

Furthermore,

let $u_2(x) = \sum[g(x_k) - Q(D)u_1(x_k)]\frac{1}{h^p}\psi(\frac{x-x_k}{h^p})\Delta_k$,

that $Q(D)(u_1(x) + u_2(x)) \sim g(x)$, and satisfy $P(D)u_2(x) = 0$.

Take the condition of $P(D)\psi(x) = 0$ into account,

finally $u_1(x) + u_2(x) \sim u(x)$.

An example for ordinary differential equation

Example:

$P(D)u(x) = f(x)$ and with some extra initial or boundary condition $\{L_j u(x) = g_j\}_{j=1}^n$, where n is the order of the equation (or degree of $P(\cdot)$).

Construction of the generator

Let $t_0, \cdots, t_{2n-1}, t_{2n}, \cdots, t_{2n+m-1}$
be the roots of $P(t)P(-t)t^m = 0$, for some positive integer m.
Denote $\Phi(x) = [t_0, \cdots, t_{2n-1}, t_{2n}, \cdots, t_{2n+m-1}]e^{tx}$ the divided difference of e^{tx} with respect to the t variable.
Let $\Phi_{\pm}(x) = (-1)^n \text{sign}(x)\Phi(x)/2$.
Then we can prove $P(D)P(-D)D^m\Phi_{\pm}(x) = \delta(x)$.
More precisely, for every function $u(x)$,

$$P(D)\int_R [P(D)u(s)][P_s(D)D^m\Phi_{\pm}(x-s)]ds = P(D)u(x).$$

Using divided difference $m! \cdot \nabla^m$ to instead of the D^m, and discrete the above integral (Modify the function to Riemannian integrable). then
$P(D) \sum [f(x_j)][P(-D)m! \cdot \nabla^m \Phi_\pm(x - x_j)]\Delta_j \sim f(x)$.
Furthermore we can construct $\varphi(x)$, which is a linear combination of the shifts of $P(-D)\Phi_\pm(x)$, that

$$P(D) \sum [f(x_j)][\varphi(x - x_j)]\Delta_j \sim P(D)u(x) = f(x),$$

possess an approximation of order m.

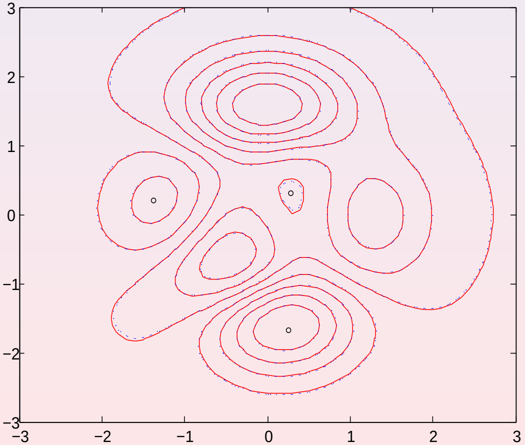
Denote $u_1(x) = \sum f(x_j)\varphi(x - x_j)$.

On the other hand, the kernel of differential operator $P(D)b(x) = 0$ is a n dimensional function space, possess a basis $\{b_j(x)\}_{j=1}^n$, and the Ansaze $u_2(x) = \sum a_j b_j(x)$ possesses an unique solution satisfying $\{L_j u_1(x) = g_j - L_j u_1\}_{j=1}^n$. Therefor

$$u^*(x) = u_1(x) + u_2(x)$$

is an approximation of $u(x)$ of order m .

Quasi interpolation for 64*64 gradine data



The Bernstein's quasi interpolation for sampling points with a random distribution

If $\{x_j^n\}$ are uniformly distributed in $[0,1]$, $W(x) \sim (1+x^2)_+^l$ or the Bernstein's polynomials, then

$$\begin{aligned} & P(|\sum f(x_j^n)W(\sqrt{n}x - j/\sqrt{n})/\sqrt{n} - f(x)| > \epsilon) \\ \leq\ & \sum_{j=0}^{n} P\{(\frac{|x_j^n - \frac{j}{n}|}{\frac{1}{\sqrt{n}}})\omega(\frac{1}{\sqrt{n}}) > \epsilon/2\} \\ \leq\ & 16/n^3\omega(\frac{1}{\sqrt{n}})^4\epsilon^4 \to 0 \ \text{ as } n \to \infty. \end{aligned}$$

Studies for random disturbs of the sampling data.

Learning

Tow kinds of the data analysis–simulation and classification.

Thanks for Your Attentions!

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Thanks for Your Attentions!